Resurrecting the New-Keynesian Model: (Un)conventional Policy and the Taylor rule

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Abstract

This paper explores the ability of the New-Keynesian (NK) model to explain the recent periods of quiet and stable inflation at near-zero nominal interest rates. We show how (conventional and unconventional) monetary policy shocks enlarge the ability to explain the facts, such that the theory supports both a negative and a positive response of inflation. Central to our finding is that monetary policy shocks may have temporary and/or permanent components. We find that the NK model can explain the recent episodes, even if one considers an active role of monetary policy and restrict ourselves to the regions of (local) determinacy. We also show that a new global solution, capturing highly nonlinear dynamics, is necessary to generate a prolonged period of near-zero interest rates as a policy choice.

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1. Introduction

> “Theories ultimately rise and fall on their ability to organize and interpret facts.” (Cochrane, 2011, p.566)

In the aftermath of the financial crisis, new-Keynesian (NK) theory has fallen on hard times. Once being a pillar of macroeconomics, in particular monetary economics, it has been criticized on both the theoretical and the empirical ends. Consider the workhorse NK model with rational expectations and active monetary policy, and the cut in interest rates from 5.25% in 2007 to 0.25% by the end of 2008 (cf. Figure 1). Interpreting this cut as an exogenous but transitory monetary policy shock, the NK model predicts a counterfactual rise of inflation to more than 4 percent. Others would argue that the fall in interest rates is a *response* to some other shock, usually to the natural rate. But then the subsequent episode of an apparently binding zero-lower bound (ZLB), which is referred to as the zero-interest-rate policy (ZIRP) period, intensified the criticism. If the economy entered a liquidity-trap scenario, the NK model would predict a deep recession with deflation (cf. Werning, 2012; Cochrane, 2017b). But nothing happened. If anything, core inflation (excluding food and energy) declined moderately to values around 1 percent in 2010. So what happened? Is the Taylor principle applicable in a world where interest rates stopped moving more than one-for-one with inflation? Cochrane (2017a) shows that alternative doctrines, including old-Keynesian models and the monetarist view, fail in explaining the ZIRP period, when the Fed drastically decreased interest rates and embarked on immense (unconventional) open market operations.

[insert Figure 1]

So the open question is on the ability of the NK model to organize our thoughts and interpret the recent facts. Is the ‘neo-Fisherian’ hypothesis, which says that inflation and interest rates are positively related, consistent with the predictions of the NK model? How can short-term interests rates be lowered to near zero values without inflation picking up as predicted by standard arguments? Despite the Fed Funds rate being kept near zero through 2015, why does CPI inflation rebound in 2011 to values around 2 percent? Central to those questions is the ability to replicate the yield curve, which from the expectation hypothesis relates to the market perceptions about future interest rates. In this paper we study whether the large cuts may have triggered changes in long-run rates.

Our contribution is to show that the ability to explain the facts crucially depends on the way we interpret and solve the model. We take a fresh look at the standard NK model under active monetary policy and show (i) that it supports both a negative and a positive response of inflation to ‘monetary policy shocks’. Our broader interpretation of monetary policy includes temporary and/or permanent components, such that a large interest rate cut may affect both ends of the yield curve. In this paper, we hypothesize that changes in constant long-run target rates can be triggered by concrete policy action, and do reflect changes in the conduct of monetary policy. Our strategy accounts for the enlarged set of policy instruments of the monetary authority. We provide economic intuition why higher interest rates might temporarily lower inflation, before eventually raising it. We also show (ii) that some counterfactual predictions are due to the approximation of the model around a zero inflation target, and can be avoided by linearizing around positive trend inflation. A nonlinear (and global) solution, which accounts for potentially highly nonlinear shock processes, performs far better in capturing non-normal times including immobile interest rates near zero and stable quiet inflation. A combination of target changes and preference shocks may result into a ZIRP period, and by the same arguments, inflation may rebound while interest rates being kept near zero values as a policy choice.

Our arguments are motivated by the strong empirical evidence of shifting end-points in the yield curve, which may just reflect the private sector's perception of the inflation target (cf. Kozicki and Tinsley, 2001; Gürkaynak, Sack, and Swanson, 2005).[1] It also relates to empirical evidence in the macro literature that the inflation target may *not* be 'anchored' but rather time-varying (cf. Ireland, 2007; Fève, Matheron, and Sahuc, 2010). Empirical evidence for the US and Japan is also confirmative of the counteracting effects resulting from transitory and permanent shocks to the interest rate (cf. Uribe, 2017). Moreover, most theorist and practitioners share a believe that with short-term interest rates kept near zero values for an extended period, an inflation target of, say, 2 percent would not carry much credibility (Eggertsson and Woodford, 2003, p.142). Hence, changes in the inflation target seems relevant for the ability of the NK model to explain the facts.

The quest on the ability of the NK model to explain the recent episodes has a deeper motivation. It sheds light on whether new theories are required to reconcile the recent facts with the theoretical predictions.[2] We aim for a parsimonious specification to organize our thoughts and interpret the data, in particular, we are interested in the necessary conditions for the NK model being capable of generating a negative short-run impact of interest rates on inflation (Cochrane, 2017a). Considering both conventional and unconventional monetary policy instruments, and accounting for nonlinearities, widely enlarges the ability to explain the data beyond standard arguments. This is true even with inflation *not* determined by the fiscal theory of the price level (FTPL), and staying in the determinacy region of the model. While the FTPL seems a promising route, we show that the minimal set of ingredients to explain the recent facts is the NK model with shocks.

In contrast to the ZLB literature, we focus on equilibria with active monetary policy,

[1]Linking the policy target rates to the long-end of the yield curve is not new and received increasing attention (see Gürkaynak and Wright, 2012, and the references therein). Time-variation in the inflation target is needed to capture the evolution of inflation expectations (cf. Del Negro and Eusepi, 2011).

[2]Del Negro, Giannoni, and Schorfheide (2015) show that the NK model with financial frictions predicts a protracted decline in inflation following a rise in financial stress around 2008.

in which the enlarged set of policy instruments includes the long-run target rates. Our simplifying assumption treats changes in the target rates as reflecting changes in the conduct of monetary policy, e.g., unconventional monetary policy. This is *not* a paper on the liquidity trap (among others Werning, 2012; Wieland, 2015; Cochrane, 2017b). We do neither claim that the ZLB is not relevant nor that the ZIRP period reflected a policy choice. This is an important empirical question, but beyond the scope of this paper. We rather try to fill the gap in the literature by providing an investigation of the NK model in non-normal times, when the simple approach fails. This is highly relevant since the mode of criticism relates to the case that the ZIRP period reflected a binding constraint. The bottom line is that changes in target rates (possibly through unconventional policies) help to explain the recent episodes within the simple NK framework, while nonlinearities play an important role to generate the ZIRP period as a policy choice.

The rest of the paper is organized as follows. First, in Section 2 we present the simple NK model and explore the ability to explain the recent facts. In Section 3 we present the full nonlinear analysis by introducing shocks, and show how near-zero interest rates can be reconciled within the framework and may result as a policy choice. Section 4 concludes. Further results and illustrations are available in an accompanying web appendix.

2. Simplified Framework

In this section we present the continuous-time specification of the standard NK model. This simple framework is used to answer our questions regarding the qualitative effect of the interest rate on inflation. In the next section we show how the equilibrium dynamics follow from the standard micro-founded rational-expectation solution and shed light on potential pitfalls when using the linearized dynamics around zero inflation targets.

The simplest version of the NK model reads:

$$\mathrm{d}x_t = (i_t - r_t - \pi_t)\mathrm{d}t \tag{1}$$

$$\mathrm{d}\pi_t = (\rho(\pi_t - \pi_{ss}) - \kappa x_t)\mathrm{d}t \tag{2}$$

This system shows the (log-)linearized equilibrium dynamics around a zero inflation target rate $\pi_{ss} = 0$. The appearance of the inflation target π_{ss} in (2) ensures that the solution of the simple model coincides with the nonlinear solution for $\pi_{ss} \neq 0$ (cf. Section A.7). Here, x_t is readily interpreted as the output gap (percentage deviations), i_t is the nominal interest rate, r_t is the 'natural interest rate' which, in the absence of shocks, coincides with the constant rate of time preference ρ, whereas κ controls the degree of price stickiness with $\kappa \to \infty$ as the frictionless (flexible price) limit, and π_t is inflation.

The equation (1) follows directly from the consumption Euler equation representing the optimal investment/saving (IS) decision, often referred to as the IS curve, whereas (2) is the NK forward-looking Phillips curve. Solving forward it expresses inflation in terms of future output gaps,

$$\pi_t - \pi_{ss} = \kappa \int_t^{\infty} e^{-\rho(k-t)} x_k \, \mathrm{d}k.$$

Hence, the *current* rate of inflation and *expected* rate of inflation are the same in continuous time. In this model it is useful to think of the path of expected future inflation and other variables (e.g., marginal cost) determining events at time t.

We close the model by specifying a rule which determines the (equilibrium) interest rates. In this perfect-foresight model both inflation dynamics and the output gap are fully determined by the Taylor rule. In what follows we analyze two alternative setups, which we refer to as the traditional feedback model:

$$i_t = \phi(\pi_t - \pi_{ss}) + i_{ss}, \quad \phi > 0, \tag{3a}$$

and the partial adjustment model (following Sims, 2004; Cochrane, 2017b):

$$\mathrm{d}i_t = \theta(\phi(\pi_t - \pi_{ss}) - (i_t - i_{ss}))\mathrm{d}t, \quad \theta > 0, \tag{3b}$$

which reflects both a response to inflation and a desire to smooth interest rates. The rules (3a) and (3b) show the attitude of the monetary authority towards either the long-run nominal interest rate or the target of inflation (one target is isomorphic to the other). In this paper, we consider the constant inflation target as a policy parameter. So we abstract from specifying a specific process, but experiment with changes in target rates, reflecting changes in the conduct of monetary policy. Empirically, variations in the target rates are crucial for understanding the dynamics of yields (cf. Bauer and Rudebusch, 2017). One potential interpretation of those changes is that economic agents infer target rates from observed interest rate and inflation dynamics: A large interest rate cut may also trigger a decrease in the long-run interest rate (or inflation) target.

The rule (3b) specifies an explicit time lag between the inflation rate π_t and the policy rate i_t. The delay will be small if the parameter θ is large.[3] While the rule (3a) may seem simpler, it has some undesirable properties in continuous time. Among others, the clear distinction between inflation (expected future inflation) that the interest rate controls and inflation that the Fed responds to vanishes in continuous time.

[3] Note that we can rewrite the partial adjustment model for $\theta > 0$ as

$$i_t - i_{ss} = \phi(\pi_t - \pi_{ss}) - (1/\theta)\, \mathrm{d}i_t / \mathrm{d}t \;\Leftrightarrow\; i_t - i_{ss} = \phi\theta \int_{-\infty}^{t} e^{-\theta(t-k)}(\pi_k - \pi_{ss}) \, \mathrm{d}k,$$

which makes i_t a *state* variable

$$i_t = \int_0^{\infty} \phi(k)\pi_{t-k} \, \mathrm{d}k.$$

2.1. Can we rule out multiple equilibria?

In this section we study local determinacy. This is relevant because the indeterminacy regions typically depend on the modelling frequency (Hintermaier, 2005). The findings for the discrete-time model with a presumed modelling frequency cannot simply be translated to different decision horizons, in particular to the continuous-time limit.

While the simple NK model with a feedback rule introduces the interest rate as a control variable, the partial adjustment model makes the interest rate a state variable, which is given by past inflation. Before we can meaningfully study shocks to the interest rate it is important to answer the question about local determinacy (focusing on local equilibria is not entirely uncontroversial, cf. Cochrane, 2011) and thus the possibility of sunspot equilibria. For the ease of presentation, we set $r_t = \rho$ in this section.

This simple NK model with a *feedback rule* has no relevant state variables. The system can be analyzed in terms of two equations (1) and (2) using (3a). A unique locally bounded solution requires two positive eigenvalues of the Jacobian matrix (cf. Appendix A.8)[4]

$$A_1 = \begin{bmatrix} 0 & \phi - 1 \\ -\kappa & \rho \end{bmatrix}.$$

Hence, a necessary (and sufficient) condition for local determinacy is $\phi > 1$. So the unique locally bounded solution is $x_t = 0$ and $\pi_t = \pi_{ss}$ such that $i_t = i_{ss} = \rho$. In other words, a negative (short-run) response of inflation to raising interest rates is not possible as long as the monetary authority implements the Taylor principle. Any monetary policy shock, which affects the policy targets, would be permanent and operates instantaneously. The response of inflation is unambiguously *positive*. In this perfect-foresight model, interest rates can be expressed in terms of future output gaps. We would also need to include a serially correlated shock in order to generate transitional dynamics in the model.

In the simple NK model with *partial adjustment*, the only relevant state variable is the interest rate (historically given inflation rates). We thus obtain the equilibrium values for the output gap and the inflation rate as policy functions $x_t = x(i_t)$ and $\pi_t = \pi(i_t)$. The system can be analyzed in terms of three equations (1), (2) and (3b), where a unique locally bounded solution requires two positive eigenvalues of the Jacobian matrix[5]

$$A_2 = \begin{bmatrix} 0 & -1 & 1 \\ -\kappa & \rho & 0 \\ 0 & \phi\theta & -\theta \end{bmatrix}.$$

[4]The Jacobian matrix has $\mathrm{tr}(A_1) = \lambda_1 + \lambda_2 = \rho > 0$ and $\det(A_1) = (\phi - 1)\kappa$ is positive for $\phi > 1$, thus both eigenvalues have positive real parts, $\lambda_1 \lambda_2 = \det(A_1)$, such that $\lambda_{1,2} = \frac{1}{2}(\rho \pm \sqrt{\rho^2 - 4((\phi-1)\kappa)})$.

[5]Note that $\det(A_2) = -\kappa\theta(\phi - 1)$ which is negative for $\phi > 1$. Further, we know that $\lambda_1 + \lambda_2 + \lambda_3 = \mathrm{tr}(A_2) = \rho - \theta$ and $\lambda_1\lambda_2\lambda_3 = \det(A_2) = -\kappa\theta(\phi - 1)$. Because a unique locally bounded solution requires two positive eigenvalues, $\phi > 1$ is necessary (and sufficient) to obtain determinacy in this model.

Again, a necessary (and sufficient) condition for local determinacy is $\phi > 1$.

Having determined the conditions for determinacy we follow the convention and define an active monetary policy if $\phi > 1$ and refer to monetary policy as passive if $\phi < 1$. One caveat is that the model is linearized around zero inflation targets. As shown below, the condition $\phi > 1$ remains necessary for $\pi_{ss} \neq 0$ (no longer sufficient though).

2.2. Which policy instruments?

The recent episodes shed light on the set of central bank instruments. In fact, the nominal interest rate, once considered as the most important (conventional) instrument, will no longer be the sole determinant of monetary policy. A large body of literature and anecdotal evidence show that unconventional policies, in particular forward guidance and quantitative easing (QE), are important monetary policy instruments too. Unless one adds financial frictions (e.g., Gertler and Karadi, 2011), or assumes imperfect substitutability between different maturities (cf. Chen, Cúrdia, and Ferrero, 2012), the NK model predicts that arbitrary QE operations are irrelevant. This is important because inflation seems to be unaffected by the large-scale asset purchase (LSAP) programmes. Hence, QE as such is *not* considered a separate policy instrument.[6] In contrast, forward guidance, which also includes the communication of the inflation target, has strong effects in the standard NK model (Del Negro, Giannoni, and Patterson, 2015; Campbell, Fisher, Justiniano, and Melosi, 2016). While the traditional policy instrument targets the short-term interest rate, the unconventional policy measures are commonly targeting interest rates at higher maturities (or the longer end of the yield curve).

Beside changing the short-term interest rate, the monetary authority may focus on other maturities, in particular the long-end of the yield curve.[7] As the (constant) inflation target is under the discretion of the monetary authority, there might be changes in its perception by economic agents due to communication and/or other policy measures. In what follows we consider the constant inflation target π_{ss} as a policy instrument.[8] In our analysis, a 'target shock' simply reflects (unexpected) changes to π_{ss}, which is interpreted to representing a different 'regime', and thus may induce transitional dynamics.

There is also an important difference with respect to forward guidance for the two Taylor rules specified in (3a) and (3b). Pure 'communication' about future policy induces a reaction of the interest rate in the feedback model due to the effect on inflation, while in the partial adjustment model interest rates are immobile on impact (pre-determined),

[6] As a caveat, LSAPs could affect term premia, a channel which is absent in the simple NK model and will be discussed later. Moreover, the LSAPs could also affect agents expectations of the future course of monetary policy (cf. Wright, 2012), which may be captured by 'shocks' to the long-run target rates.

[7] Swanson and Williams (2014) find that interest rates with a year or more to maturity were surprisingly unconstrained and responsive to news throughout 2008 to 2010.

[8] Note that the simplifying assumption of constant target rates will not be relevant for our arguments. Alternative approaches such as the regime-switching framework (see Sims and Zha, 2006), or time-varying inflation targets (e.g., Ireland, 2007) would be more realistic, at the cost of more technical details.

e.g., with respect to changes to long-run targets. So an immediate challenge for empirical research is to identify target shocks, and also to which extent an observed monetary policy shocks contain information about (perceived) changes in long-run targets.

2.3. Do higher interest rates raise or lower inflation?

Following the discussion on the policy instruments we now address the question of whether higher interest rates raise or lower inflation. In fact, the NK model for $\phi > 1$ makes sharp predictions regarding the link between interest rates and inflation, but at the same time can explain both the short-run negative response *and* the long-run positive Fisher effect (for an illustration of indeterminacy see Cochrane, 2017a). The minimal set of ingredients, in a forward-looking general equilibrium framework with active monetary policy, $\phi > 1$, to produce a negative short-run impact of interest rates on inflation is the partial adjustment model (without shocks).

[insert Figure 3]

In particular, for the partial adjustment model (and given π_{ss}), the inflation rate is a negative function of the interest rate (see Figure 3). The figure plots the initial inflation rate (or current expected inflation) for different interest rates, which shows the short-run negative relationship for given target rates. The intuition is that the interest rate depends positively on the level of inflation, but negatively on its time derivative,

$$i_t = \phi(\pi_t - \pi_{ss}) + i_{ss} - \theta^{-1}\,\mathrm{d}i_t/\,\mathrm{d}t, \quad \theta > 0. \tag{4}$$

For a given value $\mathrm{d}i_t/\,\mathrm{d}t \neq 0$, the larger the central bank's desire to smooth interest rates over time (the lower θ), the larger the second effect: Suppose that after a contractionary monetary policy shock $i_t > i_{ss}$, so the (after-shock) time-derivative of the interest rate is negative $\mathrm{d}i_t/\,\mathrm{d}t < 0$, which reflects the slope of the impulse response function. Higher interest rates are related to lower inflation rates, because the inflation rate is determined by both the (long-run) Fisher relation *and* the mean reversion back to the target level. For our parameterization, inflation falls by 0.5 percentage points on impact for an 1 percentage point increase in interest rates. To summarize, the short-run response of inflation rates on impact is *negative*, while the *positive* relationship (higher inflation targets imply higher interest rates) is still given by the long-run Fisher relation $i_t = \rho + \pi_t$.

So what happens if central banks raise interest rates? If the increase is considered by agents not only as temporary, but after all reflects a change in the bank's policy targets, inflation stability in the Fisher equation will result in higher long-run inflation. But can higher permanent interest rates reduce inflation in the short run? Indeed this is possible if the 'target shock' is accompanied by concrete policy action, i.e., a raise in the short-term interest rate. In the partial adjustment model, this induces the traditional negative effect on inflation, which may even dominate the long-run Fisher effect temporarily. However, inflation *cannot* temporarily decrease in the simple feedback model. Because of the formal peg – unless we add a persistent shock to the feedback rule – any temporary deviation from the equilibrium instantaneously jumps back. Any temporary shock would evaporate, and the interest rate will accommodate its equilibrium level (infinitely fast). Only for the case where $\theta < \infty$, any temporary change induces persistent equilibrium dynamics.

[insert Figure 5]

Let us consider a concrete example. Suppose that variables in the simple NK model are at steady state and the inflation target $\pi_{ss} = 0.02$ is lowered by 50 basis points (bp), and the short-term interest rate is *decreased* by 250 bp. Hence, the observed concrete policy action (or 'monetary policy shock') is 250 bp, but only a fraction 1/5 is permanent leaving the remainder 4/5 being temporary and not reflecting changes in policy targets. In the long run we expect lower inflation due to the Fisher relation, but temporarily the traditional negative trade-off dominates the Fisher effect (cf. Figure 5). Our simulation exercise shows that on impact the inflation rate *increases* to 2.5% and then both inflation and interest rates accommodate their new equilibrium levels after about 10 quarters.

Obviously, the presented channel not a necessary condition for a temporarily negative response of inflation. Another possibility is to add long-term debt and use the fiscal theory to pin down the price level (following McCallum, 2001; Del Negro and Sims, 2015). As shown in Cochrane (2017a), the FTPL produces a temporary reduction in the inflation rate due to the decline in the nominal market value of the debt. Adding an interest rate smoothing in the Taylor rule (3b) not only is simpler, it helps to avoid some of the undesirable properties of the feedback model in continuous time.

This new perspective on 'monetary policy shocks' offers an alternative explanation for the so-called 'prize puzzle' (going back to Sims, 1992; Eichenbaum, 1992).[9] At the risk of oversimplifying: Higher short-term interest rates (Fed Funds rate) decrease inflation, whereas higher long-run interest rates (inflation target) increase inflation.[10]

2.4. Can we explain the recent episodes?

In this section, we provide anecdotal evidence to study the ability of the simple NK model to explain the recent episodes during the new century, including the financial crisis episodes. First, to the unaided eye, the data suggests a reversal of the usual negative tradeoff in the period 2001-2007, rather supporting the 'neo-Fisherian' hypothesis, which

[9] Similarly, a cost-channel in addition to the demand channel is likely to generate a positive response on impact, but has little empirical support (see Castelnuovo, 2012, and the references therein). Castelnuovo and Surico (2010) show that accounting for *expected* inflation may also explain the 'puzzle'.

[10] Recent empirical work estimates that inflation reaches its long-run level within a year (Uribe, 2017), which is confirmative to our simulation results (cf. Figures C.3 and C.4).

says that inflation and interest rates are positively related. If anything, inflation indeed decreased in response to the interest rate cuts. Second, in the subsequent period from 2007 the Fed Funds rate has remained near zero until the liftoff in December 2015, to which we refer to as the zero-interest-rate policy (ZIRP) period, but surprisingly, inflation rates kept stable and quiet (cf. Cochrane, 2017a). Third, despite interest rates near zero through 2015, inflation rebounded already in 2011, with about the same pattern as before. Fourth, while the short rate seems immobile over the recent episode, the longer end of the yield curve has considerable variation and is declining over time. From the expectation hypothesis, we may read this as changes of market perceptions about future interest rates. Between 2004 and 2005, the federal funds rate increased by 150 bp, while the ten-year yield fell by about 70 bp (cf. Backus and Wright, 2007). So can we explain the recent episodes and term structure anomalies within the NK model?

We use an impulse response analysis and try to reconcile the most prominent features of the data discussed in the literature, which intensified the criticism, but keeping the working hypothesis of an active monetary policy ($\phi > 1$). With this hypothesis we study whether the data really is telling us that something is wrong with our models – or whether the recent episodes can be explained within the *standard* NK framework. This is important at least for two reasons: First, the traditional approach (in normal times) presumes an active monetary policy, which seems violated during the ZIRP period. So shall we abandon the celebrated Taylor rule only because of an apparently binding constraint? Second, the implications of passive monetary policy ($\phi < 1$), including the case of $\phi = 0$, are studied thoroughly in Cochrane (2017a).[11] As nicely illustrated, the problem of indeterminacy (and excess sunspot volatility) of the workhorse model may be avoided when merging the model with the FTPL. So we complement these results for $\phi > 1$.

The most prominent features of the data discussed in the literature are studied by looking at the respective periods. In what follows we study the ability of the NK model to interpret the episodes: (i) with an apparent sign reversal (2001-2007), (ii) including a zero-interest-rate policy (2007-2015), (iii) with an inflation rebound *and* near-zero interest rates (2011), and (iv) including an apparent term structure anomaly (2004-2005).

[insert Table 3]

2.4.1. Sign reversal

While the academic discourse about the effects of the nominal interest rate on the inflation rate has some tradition in macroeconomics, motivated by the 'price puzzle', it received public attention in 2008, when the interest rates in the US (followed by the ECB in 2014) hit essentially zero. Consider the period 2001-2007, right before the financial crisis.

[11] The interpretation of the data essentially followed the believes about monetary policy and the views which theory should be applied (an excellent discussion of alternative theories is in Cochrane, 2017a).

In Jan 2001 the Fed Funds rate was at 6 percent (5.98%), the long rate 10Y at 5 percent (5.16%). In Sep 2007 the Fed Funds rate was slightly below 5 percent (4.94%), the long rate 10Y at 4.5 percent (4.52%). In the meantime, the Fed Funds rate has been decreased and raised to and from 1 percent quite sharply. Over the same period, the core CPI inflation followed a similar $\vee$ pattern and decreased slightly from 2.5 percent (2.57%) to values around the announced target rate of 2 percent (2.10%). When the Fed Funds rate dipped at 1 percent (0.98%) in Dec 2003, inflation also had its lowest value of 1 percentage point (1.09%) with long yields at 3.5 percent (3.33%). Can we reconcile this pattern with the NK model?

If we interpret the $\vee$ pattern as two consecutive temporary monetary policy shocks, the NK model predicts that inflation should have followed a counterfactual $\wedge$ pattern.[12] A transitory (negative) monetary policy shock of 500 bp would imply inflation to increase by about 250 bp in 2003 (see Table 3).[13] This summarizes the puzzling 'sign reversal' and strikingly fails to explain the decline in long yields (cf. Figures C.9 and C.10). However, the same transitory shock together with a (negative) permanent shock of 200 bp would account for the observed pattern for inflation and would predict a decline in yields to longer maturities. Similarly, a (positive) temporary monetary policy shock of 400 bp together with restoring the announced inflation target rate in 2007 (cf. Figures C.15 and C.16) has the opposite effect and may have generated the observed $\vee$ pattern of Fed Funds, 10Y yields and core CPI inflation in the period from 2001 to 2007. To summarize, the observed pattern indeed can be reconciled with the (simple) NK model when allowing for a $\vee$ pattern to both the short-run *and* the long-run target rates.

2.4.2. ZIRP period

We next consider the zero-interest-rate policy (ZIRP) period 2007-2015, right after the begin of the financial crisis in Sep 2007 until the 'liftoff' in Dec 2015, with the end-point marking the start of the Fed's 'normalization' of monetary policy (Williamson, 2016).

In Sep 2007, the Fed Funds rate was at 5 percent (4.94%), the long rate 10Y rate at 4.5 percent (4.52%), while in Jan 2009 the Fed Funds rate was at 0.25 percent (0.15%), the long rate 10Y at 2.5 percent (2.52%) and stayed there. Over the same period, the core CPI inflation decreased from 2 percent (2.10%) to values way below the announced target rate around 0.5 percent (0.60%) in Oct 2010, and then bounced back in Aug 2011 to values around the announced target of 2 percent (1.97%). At a first glance, things look pretty much like the sharp decrease during the 2001-2003 period. This time, however, the (short-run) nominal interest rates was quite close to the ZLB and did not return to higher values for a while. Can we generate a ZIRP period within the NK model?

[12] To begin with, we neglect shocks to the natural rate (preference shocks) and let $r_t = \rho$.

[13] For details and transitional dynamics, we also refer to the figures in the accompanying web appendix.

An interest rate cut by about 475 bp, for an inflation target of 2 percent, we should have expected inflation rates of more than 4 percent. If anything, core inflation (excluding food and energy) declined from slightly above 2 percent to values around 0.5 percent in 2010, and then rebounded to 2 percent 2011. If one borrows the explanation of the 2001-2003 period, the sharp decrease may also reflect a change in the inflation target by 2 percent (the 10Y bond yield declined approximately by 200 bp). Inflation would jump only by 0.5% and then after about 2.5 years decline to zero (see Figures C.21 and C.22). This sounds reasonable. But it does *not* explain the near zero values of the interest rate. Although we are able to replicate the observed variables on impact, and inflation rates eventually approaching zero, the model predicts a strong tendency of the interest rate to revert back to its steady-state value. Extending the simple NK model by a shock to the 'natural rate' (preference shock), we may add a preference shock:

$$\mathrm{d}\hat{d}_t \quad = \quad -\rho_d \hat{d}_t \mathrm{d}t, \tag{5}$$

and consider $r_t \equiv (\rho + \rho_d \hat{d}_t)$ as the 'natural rate of interest' (e.g., Werning, 2012), which may be interpreted as an alternative way of reconciling the NK model prediction with the data. Is it possible that a 'preference shock', or a series of shocks to the natural rate kept interest rates at near zero values, such that our working hypothesis is fulfilled?

In fact, we may replicate both inflation and interest rates by reverse-engineering the latent process for the (realized) preference shocks. Economically we are looking for the natural rate, which implies both inflation and interest rates in the data. Everything that explains the changes in the required natural rate can potentially explain the empirical patterns. Both the inflation target rate and the preference shock affects the natural rate. So is this really an alternative explanation? This is important as we are interested in the minimum ingredients of the NK model to explain the facts.

[insert Figure 2]

Figure 2 sheds light on the natural rate required for the simple model to explain the data (cf. Figure C.41 for the required preference shock). The most interesting part is the model-implied longer-end of the yield curve. If this prediction was line with the data, the model indeed has something to say. For example, the $\vee$ pattern of Fed Funds, 10Y yields and core CPI inflation in the period 2001-2007 could have been generated by a similar pattern of the natural rate, while keeping the inflation target at 2 percent. The ZIRP period is different though. Starting around 2011, when the inflation rates rebound to higher values, the model-implied 10 year yields (Implied 10Y Govt) are substantially higher than in the data. So the simple NK model is unable to explain the longer end even for the hypothetical scenario of a series of shocks to the natural rate for a given inflation target rate. The large (negative) preference shock about 10 percent would be required to remain idle around this value to generate the start of the ZIRP period.

What we learn from the hypothetical natural rate in Figure 2 is that a series of shocks may indeed generate the observed pattern in the interest rates and inflation, but ultimately this seems inconsistent with the underlying shock process. To explain the ZIRP episode with a *single* shock we would need to modify the shock dynamics. Our simulation results confirm this conjecture: Adding a negative preference shock of roughly 10 percent to both monetary policy shocks helps to fix the yield curve and inflation, but does not generate a ZIRP period. Even with higher persistence the assumed shock process (5) would *not* imply that interest rates do remain close to zero (cf. Figures C.27 and C.28).

The bottom line of this section is that both temporary and permanent shocks are required for the ability of the simple NK model with $\phi > 1$ to explain the facts. One subtle but important issue is that such a hypothetical series of shocks seems inconsistent with the underlying shock process (5) to generate Figure 2. Perhaps, a nonlinear approach is needed to capture potentially important shock dynamics? So this is something that ultimately should be addressed, before we can safely ignore this possibility.

2.4.3. Inflation rebound (near-zero interest rates)

Elaborating on the previous results, it gets even more challenging to reconcile the facts with the simple NK model, when we consider that after the long decline since 2007, the begin of the ZIRP period, inflation suddenly rebounds to levels around 2 percent in 2011, while interest rate remain immobile and near zero at least throughout 2015.

Let us turn to the facts. In Oct 2010, the Fed Funds rate was at 0.25 percent (0.19%), the long rate 10Y at 2.5 percent (2.54%), and core CPI inflation dipped at 0.5 percent (0.60%), while in Jun 2011 the Fed Funds rate was close to zero (0.09%), whereas the long rate 10Y was at 3 percent (3.00%) and core CPI inflation increase to 1.5 percent (1.58%), with tendency to revert back to values around the 'official' target rate at 2 percent.

If we considered a target shock of 200 bp, so we re-established the announced target rate around 2011, while interest rates at near zero values, the NK model could explain the inflation rebound without any effects on nominal interest rates *on impact* with the partial adjustment model only (the feedback model cannot generate this for $\phi > 1$). If anything, the Fed Funds rate in fact was lowered by 10 bp, which seems too small to account for the rebound of inflation rates. As shown above, a shock to the natural rate may also affect the inflation rates. Hence, we consider a simultaneous negative preference shock of about 15 percent (cf. Figure C.41). Without this shock to the natural rate, inflation would jump to values around 4.5 percent for the presumed inflation target shock. A similar (counterfactual) picture arose at the longer end of the yield curve, if we would try to match the inflation figures with some preference shock only. So from our previous discussion, one plausible scenario is that the economy (still) experiences a negative natural interest rate, which keeps inflation at reasonable levels after the target shock, but lets inflation rebound with interest rates being at near zero values (cf. Table 3). Nonetheless, though the combination of target shock and preference shock may explain the rebound, and alleviates the tendency of the nominal interest rate to revert back to steady-state levels, we cannot explain the remaining ZIRP period with a single shock (cf. Figures C.33 and C.34).

So we conclude that the simple NK model fails to replicate the observed pattern in the data with a single shock. We would need a large shock to the natural rate *and* the target rate which keeps nominal interest rates at near zero values for a while and simultaneously depresses inflation. A higher inflation target, potentially triggered by market expectations of higher future inflation due to the QE operations, may have caused inflation eventually to pick up without affecting the nominal interest rate.

2.4.4. Term structure anomalies

The discussion on preference shocks vs. target shocks has shown that it is important to consider both, the short and the longer end of the yield curve in order to interpret the data through theoretical arguments. We now present some anecdotal evidence that shocks may indeed arise simultaneously. If a monetary policy shock is accompanied by a preference shock, some of the 'anomalies' observed in the date arise in the standard NK model.

Let us consider the period between 2004 and 2005, when a rotation in the yield curves gave rise to what Alan Greenspan's called a 'conundrum' (cf. Backus and Wright, 2007). In Jun 2004, the Fed Funds rate was at 1 percent (1.03%), the long rate 10Y at 4.5 percent (4.73%), while in Feb 2005, the Fed Funds rate was at 2.5 percent (2.50%), the long rate 10Y at 4 percent (4.17%). Over the same period, core CPI inflation increased from slightly below its announced target rate of 2 percent (1.87%) to about 2.5 percent (2.31%).

The 'conundrum' is that the federal funds rate increased by 150 bp, but the ten-year yield decreased by about 50 bp. Can we reconcile the rotation of the yields, to which we refer as term structure 'anomalies', with the standard NK model? Given the previous discussion, we may conjecture that a positive monetary policy shock was accompanied by a negative shock to the natural rate, keeping the inflation target about the same level, such that the standard negative relationship between interest rates and inflation is as expected (with a tendency to revert back to the target rate).[14]

We simulate a positive monetary policy shock of 150 bp which is accompanied by a negative preference shock of about 10 percent (in accordance with Figure 2). Both shocks generate the rotation in the yields as observed in the data (cf. Figures C.39 and C.40). While a rotation in the nominal yield curve could also be obtained by a contemporaneous negative target shock, two reasons speak against this hypothesis for the period 2004-2005: First, if anything, we would expect that a *rise* in nominal interest rates may trigger a *rise*

[14] This observation may give a hint that restoring the inflation target rate from 0% to 2% should have occurred already right after Dec 2003 through Jun 2004 (see the discussion in Section 2.4.1).

in the target rate. Second, the predicted real yield curve would not show a rotation as observed in the data (compare to Fig. 4 in Backus and Wright, 2007). Hence, a rotation in both yield curves suggests that the monetary policy shocks was accompanied by a negative shock to the natural rate. This results points towards directions for empirical research how we may separate target shocks from preference shocks: In order to identify shocks, we eventually employ data on both the nominal *and* the real yield curve. Though the simple model is able to account for the rotation in the yields, it is too sensitive with respect to the inflation rate (cf. Table 3). The simulated response to the monetary policy shock is a counterfactual decline. We will show below that the strong effects on the inflation rate indeed is an artefact of the linear model around zero inflation targets.

2.5. Discussion and open questions

So the bottom line is a partial remedy of the NK model to interpret the data. We show how target shocks (loosely interpreted as forward guidance) in addition to the traditional policy instrument improves the ability of the model to explain the facts. The linear model (with $\phi > 1$) is unable to account for nonlinearities, so the paper does not stop here.

We showed that the simple model still helps us to organize our thoughts, so abandon the model might be too shortsighted: Allowing monetary policy shocks to have transitory and permanent components, we may explain the sign reversal observed in the data and the $\vee$ pattern of the Fed Funds rate *and* core CPI inflation in the data. These predictions are also in line with the predictions for the yield curve. Extending the model by (transitory) shocks to the natural rate helps to explain the begin of the ZIRP period and the inflation rebound in 2011, while interest rates being immobile and near zero. One open question remains because the NK model fails to replicate the ZIRP period with a single shock such that the shock dynamics are consistent with the underlying shock process. According to Figure 2 we would need a large shock to the natural rate *and* that this shock keeps the natural rate negative for a while, before eventually reverting back to its steady state.

Perhaps need to distinguish our *approach* between normal times and non-normal times, where the dynamics are different from those at the intended equilibrium point? This is what we learn from Brunnermeier and Sannikov (2014): In normal times, the equilibrium system is near the steady state, where the system is resilient to most shocks near the steady state. Unusually large shocks, however, may induce completely different dynamics of macro aggregates. Once in a crisis state (non-normal times), also smaller shocks are subject to amplification. A nonlinear framework may be an alternative interpretation in which a single preference shock accounts for the ZIRP period. In what follows, we set up a parsimonious model, where the dynamics of large negative shocks are different from those around the steady state, at which the model is observational equivalent to (5).

Moreover, note that the presented model so far is an approximation of the NK model around zero inflation target rates, $\pi_{ss} = 0$. As we show below, the stability properties are sensitive to non-zero inflation targets, $\pi_{ss} > 0$. In particular, allowing for (even small) positive inflation targets, we need to extend the continuous-time feedback model, e.g, by an output response, to obtain a unique locally bounded solution. On the other hand, the partial adjustment model does generate local determinacy for plausible non-zero inflation targets (including the 'official' target rate). The non-zero inflation targets, however, are important to evaluate the effects of possibly shifting perceptions of the central bank's inflation target (see Gürkaynak and Wright, 2012, and the references therein).

Another caveat is that price dispersion typically is dropped as it becomes an exogenous process in the linear model around zero inflation targets. It is easy to show, however, that the price dispersion is quite different for an inflation target of about 2 percent: Higher price dispersion is associated with higher inflation. Hence, a shock to the inflation target has important implications for the price dispersion and inflation dynamics, and may also generate different dynamics for macro aggregates. It will depend on the particular question whether this is an important caveat. It might be irrelevant for small shocks, but becomes important in non-normal times, when the economy is far away from its steady state.

In what follows we thus set up and solve the nonlinear version of the NK model. We show that an alternative shock process, which is observational equivalent in normal times (with small shocks), has quiet different dynamics in non-normal times (large shocks). We also allow for stochastic shocks and show how uncertainty shocks will affect the long-end of the yield curve even if the inflation target rate is constant. A time-varying inflation target thus may be interpreted as a short cut to include 'uncertainty' shocks.

3. Nonlinear New-Keynesian Model with Shocks

We describe now the environment for our investigation. It is the continuous-time version of the standard NK model (cf. Woodford, 2003). We summarize the equilibrium dynamics, show how to compute impulse responses, compute the effects of uncertainty, and how to solve the model in the policy function space. Throughout the paper, we keep the nonlinear structure of the model, which turns out to be quite relevant for non-normal times when considering large shocks and/or large deviations from the point of approximation.

3.1. The Model

The basic structure of the model is as follows. A representative household consumes, saves, and supplies labor. The final output is assembled by a final good producer, who uses as inputs a continuum of intermediate goods manufactured by monopolistic competitors. The intermediate good producers rent labor to manufacture their good, and face the constraint that they can only adjust the price following Calvo's pricing rule (Calvo, 1983). Finally, there is a monetary authority that fixes the short-term nominal interest rate through open market operations with public debt, and a fiscal authority that taxes and consumes. We introduce four stochastic shocks, one to preferences (which can be loosely interpreted as a shock to aggregate demand, temporarily affecting the real interest rate), one to technology (interpreted as a shock to aggregate supply), one to monetary policy, and one to fiscal policy. For simplicity, we do not explicitly model a shock to the inflation target, which is considered a policy instrument under the discretion of the central bank.

3.1.1. Households

There is a representative household in the economy that maximizes the following lifetime utility function, which is separable in consumption, c_t and hours worked, l_t:

$$\mathbb{E}_0 \int_0^\infty e^{-\rho t} d_t \left\{ \log c_t - \psi \frac{l_t^{1+\vartheta}}{1+\vartheta} \right\} \mathrm{d}t, \quad \psi > 0, \tag{6}$$

where ρ is the subjective rate of time preference, ϑ is the inverse of Frisch labor supply elasticity, and d_t is a preference shock, with $\log d_t$ following an Ornstein-Uhlenbeck (OU) process (the continuous time analog of a first-order autoregression):

$$\mathrm{d} \log d_t = -\rho_d \log d_t \mathrm{d}t + \sigma_d \mathrm{d}B_{d,t}. \tag{7}$$

The process $B_{d,t}$ is a standard Brownian motion, such that by Itô's lemma:

$$\mathrm{d}d_t = -\left(\rho_d \log d_t - \tfrac{1}{2}\sigma_d^2\right) d_t \mathrm{d}t + \sigma_d d_t \mathrm{d}B_{d,t}.$$

Below, for this shock and the other exogenous stochastic processes, we will use both the formulation in level and in logs depending on the context and ease of notation.

Let a_t denote real financial wealth, the household's real wealth evolves according to:

$$\mathrm{d}a_t = \left((i_t - \pi_t)a_t - c_t + w_t l_t + T_t + \digamma_t\right) \mathrm{d}t, \tag{8}$$

in which i_t is the nominal interest rate on government bonds, π_t the rate of inflation of the price level p_t (or price of the consumption good), w_t is the real wage, T_t is a lump-sum transfer, and $\digamma_t$ are the profits of the firms in the economy.

3.1.2. The Final Good Producer

There is one final good produced using intermediate goods with the following production function:

$$y_t = \left(\int_0^1 y_{it}^{\frac{\varepsilon-1}{\varepsilon}} \mathrm{d}i \right)^{\frac{\varepsilon}{\varepsilon-1}}, \tag{9}$$

where ε is the elasticity of substitution.

Final good producers are perfectly competitive and maximize profits subject to the production function (9), taking as given all intermediate goods prices p_{it} and the final good price p_t. Hence, the input demand functions associated with this problem are:

$$y_{it} = \left(\frac{p_{it}}{p_t}\right)^{-\varepsilon} y_t \qquad \forall i,$$

and

$$p_t = \left(\int_0^1 p_{it}^{1-\varepsilon} \mathrm{d}i\right)^{\frac{1}{1-\varepsilon}} \tag{10}$$

is the (aggregate) price level.

3.1.3. Intermediate Good Producers

Each intermediate firm produces differentiated goods out of labor using:

$$y_{it} = A_t l_{it},$$

where l_{it} is the amount of the labor input rented by the firm and where A_t follows:

$$\mathrm{d}\log A_t = -\rho_A \log A_t \mathrm{d}t + \sigma_A \mathrm{d}B_{A,t}. \tag{11}$$

Therefore, the marginal cost of the intermediate good producer is the same across firms:

$$mc_t = w_t / A_t.$$

The monopolistic firms engage in infrequent price setting á la Calvo. We assume that intermediate good producers reoptimize their prices p_{it} only when a price-change signal is received. The probability (density) of receiving such a signal h periods from today is assumed to be independent of the last time the firm got the signal, and to be given by:

$$\delta e^{-\delta h}, \quad \delta > 0.$$

Thus $e^{-\delta(\tau-t)}$ denotes the probability of not having received a signal during $\tau - t$,

$$1 - \int_t^\tau \delta e^{-\delta(h-t)}\,\mathrm{d}h = 1 - \left(-e^{-\delta(\tau-t)} + 1\right) = e^{-\delta(\tau-t)}. \tag{12}$$

A fraction of firms will receive the price-change signal per unit of time. All other firms keep their old prices. Note that the lower the parameter δ, the higher price rigidities. In the frictionless case $\delta \to \infty$. Prices are set to maximize the expected discounted profits:

$$\max_{p_{it}} \mathbb{E}_t \int_t^\infty \frac{\lambda_\tau}{\lambda_t} e^{-(\rho+\delta)(\tau-t)} \left(\frac{p_{it}}{p_\tau} y_{i\tau} - mc_\tau y_{i\tau} \right) \mathrm{d}\tau \quad \text{s.t. } y_{i\tau} = \left(\frac{p_{it}}{p_\tau} \right)^{-\varepsilon} y_\tau,$$

where λ_τ is the current value (not discounted) of a unit of consumption in period τ from the perspective of the household (the pricing kernel for the firm).

After dropping constants, we may write the first-order condition as:

$$p_{it} x_{1,t} = \frac{\varepsilon}{\varepsilon - 1} p_t x_{2,t} \quad \Rightarrow \quad \Pi_t^* = \frac{\varepsilon}{\varepsilon - 1} \frac{x_{2,t}}{x_{1,t}}, \tag{13}$$

in which $\Pi_t^* \equiv p_{it}/p_t$ is the ratio between the optimal new price (common across all firms that can reset their prices) and the price of the final good and where we define the auxiliary variables (interpreted as expected discounted marginal revenue and marginal costs):

$$x_{1,t} \equiv \mathbb{E}_t \int_t^\infty \lambda_\tau e^{-(\rho+\delta)(\tau-t)} \left(\frac{p_t}{p_\tau} \right)^{1-\varepsilon} y_\tau \mathrm{d}\tau, \tag{14}$$

$$x_{2,t} \equiv \mathbb{E}_t \int_t^\infty \lambda_\tau e^{-(\rho+\delta)(\tau-t)} mc_\tau \left(\frac{p_t}{p_\tau} \right)^{-\varepsilon} y_\tau \mathrm{d}\tau. \tag{15}$$

Both variables are forward looking (or jump variables) and determined in equilibrium.

Differentiating $x_{1,t}$ with respect to time gives:

$$\mathrm{d}x_{1,t} = \left((\rho + \delta + (1-\varepsilon)\pi_t)\, x_{1,t} - \lambda_t y_t \right) \mathrm{d}t, \tag{16}$$

in which the rate of inflation $\pi_t = \mathrm{d}p_t/p_t$. Accordingly:

$$\mathrm{d}x_{2,t} = \left((\rho + \delta - \varepsilon\pi_t)\, x_{2,t} - \lambda_t mc_t y_t \right) \mathrm{d}t. \tag{17}$$

Assuming that the price-change is stochastically independent across firms gives:

$$p_t^{1-\varepsilon} = \int_{-\infty}^t \delta e^{-\delta(t-\tau)} p_{i\tau}^{1-\varepsilon} \, \mathrm{d}\tau,$$

making the price level p_t a predetermined variable at time t, its level being given by past price quotations (Calvo's insight). Differentiating with respect to time gives:

$$\mathrm{d}p_t^{1-\varepsilon} = \left(\delta p_{it}^{1-\varepsilon} - \delta \int_{-\infty}^t \delta e^{-\delta(t-\tau)} p_{i\tau}^{1-\varepsilon} \mathrm{d}\tau \right) \mathrm{d}t = \delta \left(p_{it}^{1-\varepsilon} - p_t^{1-\varepsilon} \right) \mathrm{d}t$$

and

$$\frac{1}{\mathrm{d}t} \mathrm{d}p_t^{1-\varepsilon} = (1-\varepsilon)\, p_t^{-\varepsilon} \frac{\mathrm{d}p_t}{\mathrm{d}t}.$$

Then

$$\mathrm{d}p_t = \frac{\delta}{1-\varepsilon}\left(p_{it}^{1-\varepsilon}p_t^{\varepsilon} - p_t\right)\mathrm{d}t \quad \Rightarrow \quad \pi_t = \frac{\delta}{1-\varepsilon}\left((\Pi_t^*)^{1-\varepsilon} - 1\right). \tag{18}$$

Differentiating (18) with respect to time, we obtain the inflation dynamics as:

$$\begin{aligned} \mathrm{d}\pi_t &= -\delta\left(\Pi_t^*\right)^{1-\varepsilon}\left(\pi_t + \left(mc_t/x_{2,t} - 1/x_{1,t}\right)\lambda_t y_t\right)\mathrm{d}t \\ &= -(\delta + (1-\varepsilon)\pi_t)\left(\pi_t + \left(mc_t/x_{2,t} - 1/x_{1,t}\right)\lambda_t y_t\right)\mathrm{d}t, \end{aligned} \tag{19}$$

which is interpreted as the NK Phillips-curve.

3.1.4. The Government Problem

We assume that the government sets the nominal interest rate i_t through open market operations according to two alternative setups, i.e., the feedback model:

$$i_t - i_{ss} = \phi_\pi(\pi_t - \pi_{ss}) + \phi_y(y_t/y_{ss} - 1), \quad \phi_\pi > 0,\ \phi_y \geq 0, \tag{20a}$$

or the partial adjustment model:

$$\mathrm{d}i_t = \theta(\phi_\pi(\pi_t - \pi_{ss}) + \phi_y(y_t/y_{ss} - 1) - (i_t - i_{ss}))\mathrm{d}t + \sigma_m \mathrm{d}B_{m,t}, \quad \theta > 0, \tag{20b}$$

which includes a response to inflation and output, and a desire to smooth interest rates. Similar to equation (3b), the rule in (20b) specifies a time lag between the inflation rate and the interest rate, and allows for an output response and monetary policy shocks.[15]

The coupon payments of the government perpetuities $T_t^b = -i_t a_t$ are financed through lump-sum taxes. Suppose transfers finance a given stream of government consumption expressed in terms of its constant share of output, $s_g s_{g,t}$, with a mean s_g and a stochastic component $s_{g,t}$ that follows another Ornstein-Uhlenbeck process[16]:

$$\mathrm{d}\log s_{g,t} = -\rho_g \log s_{g,t}\mathrm{d}t + \sigma_g \mathrm{d}B_{g,t}, \tag{21}$$

such that

$$g_t - T_t^b = s_g s_{g,t} y_t - T_t^b \equiv -T_t.$$

3.1.5. Aggregation

First, we derive an expression for aggregate demand:

$$y_t = c_t + g_t.$$

[15] Given our previous discussion, we will mainly focus on the partial adjustment model. Nevertheless, for the ease of comparison with the literature, we highlight some of the results for the feedback model.

[16] While we could have $s_g s_{g,t} > 1$, our calibration of s_g and σ_g is such that this event will happen with a negligibly small probability. Alternatively we could specify a stochastic process with support (0,1).

In other words, there is no possibility to transfer the output good intertemporally. With this value, the demand for each intermediate good producer is

$$y_{it} = (c_t + g_t) \left(\frac{p_{it}}{p_t} \right)^{-\varepsilon} \qquad \forall i. \tag{22}$$

Using the production function we may write:

$$A_t l_{it} = (c_t + g_t) \left(\frac{p_{it}}{p_t} \right)^{-\varepsilon}.$$

We integrate both sides:

$$A_t \int_0^1 l_{it} \mathrm{d}i = (c_t + g_t) \int_0^1 \left(\frac{p_{it}}{p_t} \right)^{-\varepsilon} \mathrm{d}i$$

to get an expression:

$$c_t + g_t = y_t = \frac{A_t}{v_t} l_t,$$

in which we define:

$$v_t = \int_0^1 \left(\frac{p_{it}}{p_t} \right)^{-\varepsilon} \mathrm{d}i$$

as the aggregate loss of efficiency induced by price dispersion of the intermediate goods. Similar to the price level, v_t is a predetermined variable (Calvo's insight):

$$v_t = \int_{-\infty}^{t} \delta e^{-\delta(t-\tau)} \left(\frac{p_{i\tau}}{p_t} \right)^{-\varepsilon} \mathrm{d}\tau. \tag{23}$$

Differentiating this expression with respect to time gives:

$$\mathrm{d}v_t \quad = \quad \left(\delta \left(\Pi_t^* \right)^{-\varepsilon} + (\varepsilon \pi_t - \delta) v_t \right) \mathrm{d}t. \tag{24}$$

Finally, as shown in the appendix, in equilibrium aggregate profits can be written as a function of other variates:

$$\digamma_t = (1 - mc_t v_t) y_t. \tag{25}$$

3.2. The HJB Equation First-Order Conditions

Define the state space $U_z \subseteq \mathbb{R}^n$ and the control region $U_x \subseteq \mathbb{R}^m$. It is also convenient to define the *reward function* $f : U_z \times U_x \rightarrow \mathbb{R}$, the *drift function* $g : U_z \times U_x \rightarrow \mathbb{R}^n$, the *diffusion function* $\sigma : U_z \rightarrow \mathbb{R}^{n \times n}$ (we discuss the details below).

Given our description of the problem, we define the household's value function as:

$$V(\mathbb{Z}_0; \mathbb{Y}_0) \equiv \max_{\{\mathbb{X}_t\}_{t=0}^{\infty}} \mathbb{E}_0 \int_0^{\infty} e^{-\rho t} f(\mathbb{Z}_t, \mathbb{X}_t) \, \mathrm{d}t,$$

in which $\mathbb{Z}_t \in U_z$ denotes the n-vector of states, $\mathbb{X}_t \in U_x$ denotes the m-vector of controls, and $\mathbb{Y}_t = \mathbb{Y}(\mathbb{Z}_t)$ is a vector of variates to be determined in equilibrium as a function of the state variables, but taken as parametric by the representative household,

$$s.t. \quad \mathrm{d}\mathbb{Z}_t = g(\mathbb{Z}_t, \mathbb{X}_t; \mathbb{Y}_t)\,\mathrm{d}t + \sigma(\mathbb{Z}_t)\,\mathrm{d}B_t,$$

where B_t is an k-vector of k independent standard Brownian motions. The instantaneous covariance matrix of $\mathbb{Z}_t$ is $\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top$, which may be less than full rank n.

In particular, the vector of state variables is $\mathbb{Z}_t = (a_t, i_t, v_t, d_t, A_t, s_{g,t})^\top$ and equilibrium variables $\mathbb{Y}_t = (y_t, mc_t, w_t, \pi_t, x_{1,t}, x_{2,t}, \Pi_t^*, \lambda_t, T_t, F_t)^\top$ to be determined endogenously, and $\mathbb{X}_t = (c_t, l_t)^\top$ is the vector of controls. In our case, the *reward function* reads:

$$f(\mathbb{Z}_t, \mathbb{X}_t) = d_t \log c_t - d_t \psi \frac{l_t^{1+\vartheta}}{1+\vartheta}.$$

From the discussion above, we define the *drift function* (with partial adjustment):

$$g(\mathbb{Z}_t, \mathbb{X}_t; \mathbb{Y}_t) = \begin{bmatrix} (i_t - \pi_t)a_t - c_t + w_t l_t + T_t + F_t \\ \theta\phi_\pi(\pi_t - \pi_{ss}) + \theta\phi_y(y_t/y_{ss} - 1) - \theta(i_t - i_{ss}) \\ \delta\,(\Pi_t^*)^{-\varepsilon} + (\varepsilon\pi_t - \delta)v_t \\ -(\rho_d \log d_t - \frac{1}{2}\sigma_d^2)d_t \\ -(\rho_A \log A_t - \frac{1}{2}\sigma_A^2)A_t \\ -(\rho_g \log s_{g,t} - \frac{1}{2}\sigma_g^2)s_{g,t} \end{bmatrix} \begin{matrix} (8) \\ (20b) \\ (24) \\ (7) \\ (11) \\ (21) \end{matrix},$$

and the *diffusion function* of the state transition equations:

$$\sigma(\mathbb{Z}_t) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma_m & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_d d_t & 0 & 0 \\ 0 & 0 & 0 & 0 & \sigma_A A_t & 0 \\ 0 & 0 & 0 & 0 & 0 & \sigma_g s_{g,t} \end{bmatrix} \begin{matrix} (8) \\ (20b) \\ (24) \\ (7) \\ (11) \\ (21) \end{matrix}.$$

By choosing the control $\mathbb{X}_t \in \mathbb{R}_+^2$ at time t, the HJB equation reads:

$$\rho V(\mathbb{Z}_t; \mathbb{Y}_t) = \max_{\{\mathbb{X}_t\}_{t=0}^\infty} \left\{ f(\mathbb{Z}_t, \mathbb{X}_t) + g(\mathbb{Z}_t, \mathbb{X}_t; \mathbb{Y}_t)^\top V_\mathbb{Z} + \tfrac{1}{2}\mathrm{tr}\left(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top V_{\mathbb{Z}\mathbb{Z}}\right) \right\}, \tag{26}$$

where $V_\mathbb{Z}$ is an n-vector, $V_{\mathbb{Z}\mathbb{Z}}$ is a $n \times n$ matrix, and $\mathrm{tr}(\cdot)$ denotes the trace of a matrix. A neat result about the formulation of our problem in continuous time is that the HJB equation (26) is, in effect, a deterministic functional equation. In the discrete-time version, we need to numerically approximate expectations (or the n-dimensional integral).

The first-order conditions with respect to c_t and l_t for any interior solution are:

$$\frac{d_t}{c_t} = V_a, \tag{27}$$
$$d_t \psi l_t^{\vartheta} = V_a w_t, \tag{28}$$

or, eliminating the costate variable (for $\psi \neq 0$):

$$\psi l_t^{\vartheta} c_t = w_t,$$

which is the standard static optimality condition between labor and consumption.

Most notably, the first-order conditions (27) and (28) yield *optimal controls*:

$$\mathbb{X}_t = \mathbb{X}(\mathbb{Z}_t, V_{\mathbb{Z}}(\mathbb{Z}_t; \mathbb{Y}_t); \mathbb{Y}_t) \equiv \begin{bmatrix} c(\mathbb{Z}_t, V_{\mathbb{Z}}(\mathbb{Z}_t; \mathbb{Y}_t); \mathbb{Y}_t) \\ l(\mathbb{Z}_t, V_{\mathbb{Z}}(\mathbb{Z}_t; \mathbb{Y}_t); \mathbb{Y}_t) \end{bmatrix} = \begin{bmatrix} (V_a(\mathbb{Z}_t; \mathbb{Y}_t))^{-1} d_t \\ (V_a(\mathbb{Z}_t; \mathbb{Y}_t) w_t / (d_t \psi))^{1/\vartheta} \end{bmatrix}.$$

Thus, the first-order conditions (27) and (28) make the optimal controls functions of the states, $c_t = c(\mathbb{Z}_t; \mathbb{Y}_t)$, $l_t = l(\mathbb{Z}_t; \mathbb{Y}_t)$. Hence, the concentrated HJB equation reads:

$$\begin{aligned} \rho V(\mathbb{Z}_t; \mathbb{Y}_t) &= f(\mathbb{Z}_t, \mathbb{X}(\mathbb{Z}_t, V_{\mathbb{Z}}(\mathbb{Z}_t; \mathbb{Y}_t)) + g(\mathbb{Z}_t, \mathbb{X}(\mathbb{Z}_t, V_{\mathbb{Z}}(\mathbb{Z}_t; \mathbb{Y}_t)); \mathbb{Y})^{\top} V_{\mathbb{Z}} \\ &\quad + \tfrac{1}{2} \mathrm{tr}\left(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^{\top} V_{\mathbb{Z}\mathbb{Z}}\right). \end{aligned} \tag{29}$$

Note that $V_a(\mathbb{Z}_t; \mathbb{Y}_t) = \lambda_t$ in (27) and (28) is readily interpreted as the marginal value of wealth or the current value of a unit of consumption in period t, and thus determines the asset pricing kernel in this economy. In what follows, we provide the asset pricing kernel or the stochastic discount factor (SDF) consistent with equilibrium dynamics of macro aggregates, which can be used to price any asset in the economy.

As we show in the appendix, the marginal value of wealth evolves according to:

$$\begin{aligned} \mathrm{d}\lambda_t &= (\rho - i_t + \pi_t)\lambda_t \mathrm{d}t \\ &\quad + \sigma_d d_t \lambda_d \mathrm{d}B_{d,t} + \sigma_A A_t \lambda_A \mathrm{d}B_{A,t} + \sigma_g s_{g,t} \lambda_g \mathrm{d}B_{g,t} + \sigma_m \lambda_r \mathrm{d}B_{m,t}, \end{aligned} \tag{30}$$

which determines the equilibrium SDF (see Hansen and Scheinkmann, 2009):

$$m_s/m_t = e^{-\rho(s-t)} \frac{V_a(\mathbb{Z}_s; \mathbb{Y}_s)}{V_a(\mathbb{Z}_t; \mathbb{Y}_t)}, \quad \text{and} \quad m_t = e^{-\rho t} \lambda_t, \tag{31}$$

or, equivalently, the present value shadow price. After some algebra (see Appendix A.2), we arrive at the Euler equation, which shows the equilibrium dynamics of consumption:

$$\begin{aligned} \mathrm{d}c_t &= -(\rho - i_t + \pi_t - \sigma_A^2 \tilde{c}_A^2 - \sigma_g^2 \tilde{c}_g^2 - \sigma_m^2 \tilde{c}_r^2 + \rho_d \log d_t + (\tilde{c}_d(1 - \tilde{c}_d) - \tfrac{1}{2})\sigma_d^2) c_t \mathrm{d}t \\ &\quad + \sigma_d \tilde{c}_d c_t \mathrm{d}B_{d,t} + \sigma_A \tilde{c}_A c_t \mathrm{d}B_{A,t} + \sigma_g \tilde{c}_g c_t \mathrm{d}B_{g,t} + \sigma_m \tilde{c}_r c_t \mathrm{d}B_{m,t}, \end{aligned} \tag{32}$$

where $\tilde{c}_r \equiv c_r/c_t$, $\tilde{c}_d \equiv c_d d_t/c_t$, $\tilde{c}_g \equiv c_g s_{g,t}/c_t$, and $\tilde{c}_A \equiv c_A A_t/c_t$, reflecting the slope of the consumption function with respect to the state variables that are driven by shocks.

3.3. Equilibrium dynamics

In order to understand the recursive-competitive equilibrium (as defined in Appendix A.3), it is instructive to show the mechanics of the NK model. We start from market clearing:

$$c_t = y_t - g_t = (1 - s_g s_{g,t})y_t = (1 - s_g s_{g,t})A_t l_t/v_t, \tag{33}$$

such that the combined first-order condition reads:

$$w_t = \psi l_t^{\vartheta} c_t \;\Leftrightarrow\; v_t w_t = \psi l_t^{1+\vartheta}(1 - s_g s_{g,t})A_t \;\Leftrightarrow\; l_t^{1+\vartheta} = \frac{v_t w_t}{(1 - s_g s_{g,t})A_t \psi},$$

and from (28):

$$c_t = ((1 - s_g s_{g,t})/v_t)^{\frac{\vartheta}{1+\vartheta}} A_t (mc_t/\psi)^{\frac{1}{1+\vartheta}}, \tag{34}$$

or

$$mc_t = \psi l_t^{1+\vartheta}(1 - s_g s_{g,t})/v_t.$$

For a given level of current marginal cost, mc_t, the solution is known analytically. However, in the NK model the firm takes into account current marginal cost *and* expected future marginal cost whenever it has an opportunity to adjust its price. Hence, the equilibrium value for current marginal costs is an unknown function of all states, $\mathbb{Y}_t = \mathbb{Y}(\mathbb{Z}_t)$.

So we arrive at a system of 5 endogenous processes, i.e., for the auxiliary variables $x_{1,t}$, $x_{2,t}$, price dispersion v_t, the Taylor rule i_t, and the consumption Euler equation c_t, and 3 exogenous shock processes for $s_{g,t}, d_t, A_t$, which summarize equilibrium dynamics:

$$\begin{aligned}
\mathrm{d}c_t &= -(\rho - i_t + \pi_t - \sigma_A^2 \tilde{c}_A^2 - \sigma_g^2 \tilde{c}_g^2 - \sigma_m^2 \tilde{c}_r^2 + \rho_d \log d_t + (\tilde{c}_d(1 - \tilde{c}_d) - \tfrac{1}{2})\sigma_d^2)c_t \mathrm{d}t \\
&\quad + \sigma_d \tilde{c}_d c_t \mathrm{d}B_{d,t} + \sigma_A \tilde{c}_A c_t \mathrm{d}B_{A,t} + \sigma_g \tilde{c}_g c_t \mathrm{d}B_{g,t} + \sigma_m \tilde{c}_r c_t \mathrm{d}B_{m,t} \\
\mathrm{d}x_{1,t} &= ((\rho + \delta - (\varepsilon - 1)\pi_t)x_{1,t} - d_t/(1 - s_g s_{g,t}))\,\mathrm{d}t \\
\mathrm{d}x_{2,t} &= ((\rho + \delta - \varepsilon\pi_t)\,x_{2,t} - mc_t d_t/(1 - s_g s_{g,t}))\,\mathrm{d}t \\
\mathrm{d}i_t &= \theta(\phi_\pi(\pi_t - \pi_{ss}) + \phi_y(y_t/y_{ss} - 1) - (i_t - i_{ss}))\mathrm{d}t + \sigma_m \mathrm{d}B_{m,t} \\
\mathrm{d}v_t &= (\delta(1 + \pi_t(1 - \varepsilon)/\delta)^{-\frac{\varepsilon}{1-\varepsilon}} + (\varepsilon\pi_t - \delta)v_t)\mathrm{d}t
\end{aligned}$$

in which $(1 + \pi_t(1 - \varepsilon)/\delta)^{\frac{1}{1-\varepsilon}} = \varepsilon/(\varepsilon - 1)(x_{2,t}/x_{1,t})$ determines the inflation rate and

$$\begin{aligned}
d_t/c_t &= ((1 - s_g s_{g,t})/v_t)^{-\frac{\vartheta}{1+\vartheta}} (mc_t/\psi)^{-\frac{1}{1+\vartheta}} d_t/A_t \\
\Leftrightarrow\; mc_t &= \psi((d_t/c_t)(A_t/d_t))^{-(1+\vartheta)}(v_t/(1 - s_g s_{g,t}))^{\vartheta},
\end{aligned} \tag{35}$$

pins down marginal costs. Given a solution to the system of dynamic equations augmented by the stochastic processes (7), (11), and (21), the general equilibrium policy functions (as a function of relevant state variables) can be obtained.

3.4. Numerical solution of the (conditional) deterministic system

In what follows we solve the NK model using the (conditional) deterministic system, which demands that we need to account appropriately for risk. This is obtained if the (nonlinear) solution to the HJB equation implies the same policy function of the boundary value problem. The solution of the deterministic model is contained as a special case.

We start from the HJB equation (29) or (a more detailed version (A.8)) and find that for $V_{aa}(\mathbb{Z}_t;\mathbb{Y}_t) \neq 0$

$$\begin{aligned} c(\mathbb{Z}_t;\mathbb{Y}_t) \;=\;& (i_t-\pi_t)a_t + w_t l(\mathbb{Z}_t;\mathbb{Y}_t) + T_t + \digamma_t - (\rho-(i_t-\pi_t))\frac{V_a(\mathbb{Z}_t;\mathbb{Y}_t)}{V_{aa}(\mathbb{Z}_t;\mathbb{Y}_t)} \\ &+(\theta\phi_\pi(\pi_t-\pi_{ss})+\theta\phi_y(y_t/y_{ss}-1)-\theta(i_t-i_{ss}))\frac{V_{ra}(\mathbb{Z}_t;\mathbb{Y}_t)}{V_{aa}(\mathbb{Z}_t;\mathbb{Y}_t)} \\ &+\tfrac{1}{2}\sigma_m^2\frac{V_{rra}(\mathbb{Z}_t;\mathbb{Y}_t)}{V_{aa}(\mathbb{Z}_t;\mathbb{Y}_t)} + \left(\delta(1+(1-\varepsilon)\pi_t/\delta)^{-\frac{\varepsilon}{1-\varepsilon}} + (\varepsilon\pi_t-\delta)v_t\right)\frac{V_{va}(\mathbb{Z}_t;\mathbb{Y}_t)}{V_{aa}(\mathbb{Z}_t;\mathbb{Y}_t)} \\ &-(\rho_d\log d_t-\tfrac{1}{2}\sigma_d^2)d_t\frac{V_{da}(\mathbb{Z}_t;\mathbb{Y}_t)}{V_{aa}(\mathbb{Z}_t;\mathbb{Y}_t)} + \tfrac{1}{2}\sigma_d^2 d_t^2\frac{V_{dda}(\mathbb{Z}_t;\mathbb{Y}_t)}{V_{aa}(\mathbb{Z}_t;\mathbb{Y}_t)} \\ &-(\rho_A\log A_t-\tfrac{1}{2}\sigma_A^2)A_t\frac{V_{Aa}(\mathbb{Z}_t;\mathbb{Y}_t)}{V_{aa}(\mathbb{Z}_t;\mathbb{Y}_t)} + \tfrac{1}{2}\sigma_A^2 A_t^2\frac{V_{AAa}(\mathbb{Z}_t;\mathbb{Y}_t)}{V_{aa}(\mathbb{Z}_t;\mathbb{Y}_t)} \\ &-(\rho_g\log s_{g,t}-\tfrac{1}{2}\sigma_g^2)s_{g,t}\frac{V_{ga}(\mathbb{Z}_t;\mathbb{Y}_t)}{V_{aa}(\mathbb{Z}_t;\mathbb{Y}_t)} + \tfrac{1}{2}\sigma_g^2 s_{g,t}^2\frac{V_{gga}(\mathbb{Z}_t;\mathbb{Y}_t)}{V_{aa}(\mathbb{Z}_t;\mathbb{Y}_t)}, \end{aligned} \tag{36}$$

which we will use to define the Euler equation errors below.

In what follows, we compute the solution to the HJB equation from a deterministic system of differential equations (a boundary value problem), which works in continuous time since the HJB equation itself becomes a deterministic equation (cf. Chang, 2004).[17] Nevertheless, we have to account appropriately for risk. So the idea is to transform the system of SDEs into a system of PDEs which also solves the HJB equation. Assume the existence of a consumption function $c_t = c(\mathbb{Z}_t)$, and use Itô's formula to arrive at:

$$\begin{aligned} \mathrm{d}c_t \;=\;& c_a\,\mathrm{d}a_t + c_r\,\mathrm{d}i_t + \tfrac{1}{2}c_{rr}\sigma_m^2\,\mathrm{d}t + c_v\,\mathrm{d}v_t \\ &+c_d\,\mathrm{d}d_t + \tfrac{1}{2}c_{dd}(\sigma_d d_t)^2\mathrm{d}t + c_A\,\mathrm{d}A_t + \tfrac{1}{2}c_{AA}(\sigma_A A_t)^2\mathrm{d}t + c_g\,\mathrm{d}s_{g,t}\mathrm{d}t + \tfrac{1}{2}c_{gg}(\sigma_g s_{g,t})^2\mathrm{d}t. \end{aligned}$$

This leads us to the following proposition.

[17] In contrast, the discrete-time HJB equation requires the analyst needs to evaluate the state space not only at the current information set, but also at future expected values, so the continuous-time approach does not require to numerically compute expectations (a burdensome step in discrete-time models).

Proposition 1. *By subtracting the Itô second-order terms from the Euler equation (32),*

$$dc_t - \tfrac{1}{2}c_{rr}\sigma_m^2\,dt - \tfrac{1}{2}c_{dd}(\sigma_d d_t)^2 dt - \tfrac{1}{2}c_{AA}(\sigma_A A_t)^2 dt - \tfrac{1}{2}c_{gg}(\sigma_g s_{g,t})^2 dt =$$
$$c_a\,da_t + c_r\,di_t + c_v\,dv_t + c_A\,dA_t + c_d\,dd_t + c_g\,ds_{g,t},$$

and inserting dc_t from (32) we may eliminate time (and stochastic shocks) and together with $c_t = d_t V_a^{-1}$ yields (36) from the HJB equation.

Proof. Appendix A.4 ■

A system of PDEs which implies the same policy function is constructed using (32) and Proposition 1 by subtracting Itô terms from the Euler equation (accounting for risk) and setting $\mathrm{d}B_{d,t} = \mathrm{d}B_{A,t} = \mathrm{d}B_{g,t} = \mathrm{d}B_{m,t} = 0$ (in the absence of shocks),

$$\begin{aligned}\mathrm{d}c_t &= -(\rho - (i_t - \pi_t))c_t\mathrm{d}t + \tilde{c}_d^2\sigma_d^2 c_t\mathrm{d}t + \tilde{c}_A^2\sigma_A^2 c_t\mathrm{d}t + \tilde{c}_g^2\sigma_g^2 c_t\mathrm{d}t + \tilde{c}_r^2\sigma_m^2 c_t\mathrm{d}t\\ &\quad -\tfrac{1}{2}\tilde{c}_{dd}\sigma_d^2 c_t\mathrm{d}t - \tfrac{1}{2}\tilde{c}_{AA}\sigma_A^2 c_t\mathrm{d}t - \tfrac{1}{2}\tilde{c}_{gg}\sigma_g^2 c_t\mathrm{d}t - \tfrac{1}{2}\tilde{c}_{rr}\sigma_m^2 c_t\,\mathrm{d}t\\ &\quad -c_t\rho_d \log d_t\mathrm{d}t + \tfrac{1}{2}\sigma_d^2 c_t\mathrm{d}t - \tilde{c}_d\sigma_d^2 c_t\mathrm{d}t\end{aligned}$$

where we define $\tilde{c}_{rr} \equiv c_{rr}/c_t$, $\tilde{c}_{dd} \equiv c_{dd}d_t^2/c_t$, $\tilde{c}_{gg} \equiv c_{gg}s_{g,t}^2/c_t$, and $\tilde{c}_{AA} \equiv c_{AA}A_t^2/c_t$ reflecting curvature of the consumption function with respect to the state variables that are driven by shocks, such that $\mathrm{d}c_t = c_a\,\mathrm{d}a_t + c_r\,\mathrm{d}i_t + c_v\,\mathrm{d}v_t + c_A\,\mathrm{d}d_t + c_d\,\mathrm{d}d_t + c_g\,\mathrm{d}s_{g,t}$ solves (36). So we refer to the following system of PDEs as the *conditional* deterministic system:

$$\begin{aligned}\mathrm{d}c_t &= -(\rho - (i_t - \pi_t))c_t\mathrm{d}t + \tilde{c}_d^2\sigma_d^2 c_t\mathrm{d}t + \tilde{c}_A^2\sigma_A^2 c_t\mathrm{d}t + \tilde{c}_g^2\sigma_g^2 c_t\mathrm{d}t + \tilde{c}_r^2\sigma_m^2 c_t\mathrm{d}t\\ &\quad -\tfrac{1}{2}\tilde{c}_{dd}\sigma_d^2 c_t\mathrm{d}t - \tfrac{1}{2}\tilde{c}_{AA}\sigma_A^2 c_t\mathrm{d}t - \tfrac{1}{2}\tilde{c}_{gg}\sigma_g^2 c_t\mathrm{d}t - \tfrac{1}{2}\tilde{c}_{rr}\sigma_m^2 c_t\,\mathrm{d}t\\ &\quad -c_t\rho_d \log d_t\mathrm{d}t + \tfrac{1}{2}\sigma_d^2 c_t\mathrm{d}t - \tilde{c}_d\sigma_d^2 c_t\mathrm{d}t \qquad (37)\\ \mathrm{d}i_t &= \theta(\phi_\pi(\pi_t - \pi_{ss}) + \phi_y(y_t/y_{ss} - 1) - (i_t - i_{ss}))\,\mathrm{d}t\\ \mathrm{d}v_t &= (\delta\,(1 + (1-\varepsilon)\pi_t/\delta)^{-\frac{\varepsilon}{1-\varepsilon}} + (\varepsilon\pi_t - \delta)v_t)\,\mathrm{d}t\\ \mathrm{d}d_t &= -\left(\rho_d \log d_t - \tfrac{1}{2}\sigma_d^2\right) d_t\mathrm{d}t\\ \mathrm{d}A_t &= -\left(\rho_A \log A_t - \tfrac{1}{2}\sigma_A^2\right) A_t\mathrm{d}t\\ \mathrm{d}s_{g,t} &= -\left(\rho_g \log s_{g,t} - \tfrac{1}{2}\sigma_g^2\right) s_{g,t}\mathrm{d}t\\ \mathrm{d}x_{1,t} &= ((\rho + \delta - (\varepsilon - 1)\pi_t)x_{1,t} - d_t/(1 - s_g s_{g,t}))\,\mathrm{d}t\\ \mathrm{d}x_{2,t} &= ((\rho + \delta - \varepsilon\pi_t)\,x_{2,t} - mc_t d_t/(1 - s_g s_{g,t}))\,\mathrm{d}t\end{aligned}$$

So the Euler equation (37) of the conditional deterministic system is used to obtain the conditional deterministic (or stochastic) steady state.[18] Recall that the inflation rate π_t is endogenously determined from (18), and the jump variables $x_{1,t}$ and $x_{2,t}$. We restrict our attention to the solution which leads the economy towards the (stochastic) steady state,

[18] Though there will be a steady-state distribution, we follow the convention in the literature and define the fix point of this system as the 'stochastic steady state', and thus use both terms interchangeably.

in which $\pi_t \to \pi_{ss}$. By solving for the time paths, the solution satisfies both the initial and the transversality condition (TVC) and characterizes the stable manifold. We iterate computing controls and updating the derivatives until convergence (cf. Table 1).[19]

[insert Table 1]

It is important to note that as long as $\|(\sigma_d, \sigma_A, \sigma_g, \sigma_m)\| \neq 0$, the term $\mathrm{d}c_t$ of the conditional deterministic system (37) does *not* coincide with the term $\mathrm{d}c_t$ of the Euler equation (32), which is an abuse of notation only needed in the numerical solution. Once we derived the policy functions, the original Euler equation is used to simulate the model and/or to make statistical inference, by allowing for the arrival of stochastic shocks.

We solve the system of PDEs by the Waveform Relaxation algorithm. In this way, we can separate the solution in the time dimension from the solution in the policy space, which turns out to be computationally more robust and less expensive.[20] Following the idea in Posch and Trimborn (2013) we obtain the unknown derivatives starting from the solution of the deterministic system, then iteratively define $\tilde{c}_r(\mathbb{Z}_t;\mathbb{Y}_t) \equiv c_r/c_t$, $\tilde{c}_{rr}(\mathbb{Z}_t;\mathbb{Y}_t) \equiv c_{rr}/c_t$, $\tilde{c}_d(\mathbb{Z}_t;\mathbb{Y}_t) \equiv c_d d_t/c_t$, $\tilde{c}_{dd}(\mathbb{Z}_t;\mathbb{Y}_t) \equiv c_{dd} d_t^2/c_t$, $\tilde{c}_g(\mathbb{Z}_t;\mathbb{Y}_t) \equiv c_g s_{g,t}/c_t$, $\tilde{c}_{gg}(\mathbb{Z}_t;\mathbb{Y}_t) \equiv c_{gg} s_{g,t}^2/c_t$, $\tilde{c}_A(\mathbb{Z}_t;\mathbb{Y}_t) \equiv c_A A_t/c_t$, and $\tilde{c}_{AA}(\mathbb{Z}_t;\mathbb{Y}_t) \equiv c_{AA} A_t^2/c_t$, and solve the system of ODEs. The initial value for the control and/or jump variables is used to approximate the solution in the policy function space (using tensor products of univariate grids as initial values), then update the solution, and iterate until convergence.

In the boundary value problem (BVP) we seek a function $x : [0, T] \mapsto \mathbb{R}^k$ that satisfies the (conditional) deterministic system consisting of the Euler equation (37) determining c_t, and the law of motion for $x_{1,t}, x_{2,t}, i_t, v_t, d_t, A_t$, and $s_{g,t}$ (which gives $k = 8$), together with the given initial conditions for the states $(i_0, v_0, d_0, A_0, s_{g,0})$ and the TVC assuming that variables approach their (stochastic) steady state values. One complication is that the time horizon is infinite, so we use the following transformation of time:

$$t = \frac{\tau}{\nu(1-\tau)} \quad \text{for} \quad \tau \in [0, 1),$$

where ν is a positive (nuisance) parameter, such that for $t \to \infty$ we have that $\tau \to 1$. Alternatively, we may set T sufficiently large but finite number.[21]

3.5. Numerical solution in the policy function space

In what follows, we show how we may alternatively solve the HJB equation (29) directly by collocation based on the Matlab `CompEcon` toolbox (Miranda and Fackler 2002).

[19] It is important to note that a recursion as set out in Table 1 is only required if we are interested in the solution of the stochastic model where $\|(\sigma_d, \sigma_A, \sigma_g, \sigma_m)\| \neq 0$.

[20] It is possible to parallelize the computation by allocating the grid of state variables to workers.

[21] Trimborn, Koch, and Steger (2008) introduced the relaxation algorithm to applications in economics. In contrast to their approach, we use projection methods to solve the boundary value problem, which turns out to be relatively efficient and (even for a few approximation points) highly accurate.

Since the functional form of the solution is unknown, an alternative strategy for solving the HJB equation is to approximate $V(\mathbb{Z}_t;\mathbb{Y}_t) \approx \phi(\mathbb{Z}_t;\mathbb{Y}_t)v$, in which v is an n-vector of coefficients and ϕ is the $n \times n$ basis matrix. The computational burden can be reduced when replacing the tensor product by sparse grids (Winschel and Krätzig, 2010). Starting from the HJB equation (29), we may approximation of the value function and/or control variables for given set of collocation nodes and basis functions $\phi(\mathbb{Z}_t;\mathbb{Y}_t)$:

$$\rho\phi(\mathbb{Z}_t;\mathbb{Y}_t)v = f(\mathbb{Z}_t,\mathbb{X}_t) + g(\mathbb{Z}_t,\mathbb{X}_t)^\top \phi_{\mathbb{Z}}(\mathbb{Z}_t;\mathbb{Y}_t)v + \tfrac{1}{2}\mathrm{tr}\left(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top \phi_{\mathbb{Z}\mathbb{Z}}(\mathbb{Z}_t;\mathbb{Y}_t)v\right),$$

or

$$v = \left(\rho\phi(\mathbb{Z}_t;\mathbb{Y}_t) - g(\mathbb{Z}_t,\mathbb{X}_t)^\top \phi_{\mathbb{Z}} - \tfrac{1}{2}\mathrm{tr}\left(\sigma(\mathbb{Z}_t)\sigma(\mathbb{Z}_t)^\top \phi_{\mathbb{Z}\mathbb{Z}}\right)\right)^{-1} f(\mathbb{Z}_t,\mathbb{X}_t),$$

which yields the coefficients based on a Newton method. This approach, however, requires a good initial guess, but is extremely useful to verify whether the implied solution obtained from the conditional deterministic system indeed solves the HJB equation.

Our results confirm the findings in the literature that the effects of uncertainty in the standard NK model are small, and primarily a level shift (cf. Figures C.43 to C.51). Both the deterministic steady state and the stochastic steady state are very close. What is more important is a substantial level shift of the policy functions around equilibrium interests rates and at near-zero values for the nominal interests rates.

3.6. Impulse responses

To compute the impulse response functions (IRFs), we initialize the state variables, given the solution $V(\mathbb{Z}_t;\mathbb{Y}(\mathbb{Z}_t)) \approx \phi(\mathbb{Z}_t;\mathbb{Y}(\mathbb{Z}_t)v$, or the consumption function (36), and solve the resulting system of ODEs following Posch and Trimborn (2013). Because we use a global (and nonlinear) solution technique, in principle, we may initialize the system at any state vector. Hence, we do not need to restrict our analysis to situations, where the economy is assumed to be in the close neighborhood of the steady state (or normal times). This is particularly important since we want to study the equilibrium dynamics in a situation where the nominal interest rate is close to zero and/or the economy is hit by large shocks (non-normal times). In fact, the computed IRF is the equilibrium time path of economic variables, which reflect a single transitional path to the (stochastic) steady state.

3.7. Implied risk premium and the natural rate of interest

We approach the definition of a risk premium from the equilibrium long-run interest rate. At the (stochastic) steady state we obtain from (37) that

$$\begin{aligned}\rho \;=\;& i_{ss} - \pi_{ss} + \tilde{c}_d^2\sigma_d^2 + \tilde{c}_A^2\sigma_A^2 + \tilde{c}_g^2\sigma_g^2 + \tilde{c}_r^2\sigma_m^2 \\ & -\tfrac{1}{2}\tilde{c}_{dd}\sigma_d^2 - \tfrac{1}{2}\tilde{c}_{AA}\sigma_A^2 - \tfrac{1}{2}\tilde{c}_{gg}\sigma_g^2 - \tfrac{1}{2}\tilde{c}_{rr}\sigma_m^2 - \tilde{c}_d\sigma_d^2 - \rho_d \log d_{ss} + \tfrac{1}{2}\sigma_d^2.\end{aligned}$$

Thus the steady-state risk premium is defined based on the identity $\rho + \pi_{ss} \equiv i^f_{ss}$ or

$$\rho + \pi_{ss} = i_{ss} + (\tilde{c}_d^2 - \tfrac{1}{2}\tilde{c}_{dd} - \tilde{c}_d)\sigma_d^2 + (\tilde{c}_A^2 - \tfrac{1}{2}\tilde{c}_{AA})\sigma_A^2 + (\tilde{c}_g^2 - \tfrac{1}{2}\tilde{c}_{gg})\sigma_g^2 + (\tilde{c}_r^2 - \tfrac{1}{2}\tilde{c}_{rr})\sigma_m^2$$

in which i^f_{ss} denotes the shadow risk-free rate (or the certainty equivalent rate of return). The long-run interest rate is decomposed into $i_{ss} = i^f_{ss} + RP_{ss}$, where

$$RP_{ss} \equiv -(\tilde{c}_d^2 - \tfrac{1}{2}\tilde{c}_{dd} - \tilde{c}_d)\sigma_d^2 - (\tilde{c}_A^2 - \tfrac{1}{2}\tilde{c}_{AA})\sigma_A^2 - (\tilde{c}_g^2 - \tfrac{1}{2}\tilde{c}_{gg})\sigma_g^2 - (\tilde{c}_r^2 - \tfrac{1}{2}\tilde{c}_{rr})\sigma_m^2.$$

For our parameterization $RP_{ss} \approx 0.0695\%$, so it increases the nominal interest rate by approximately 0.07% or 7 bp. Consistently, we define the risk premium (cf. Posch, 2011):

$$\rho - \frac{1}{\mathrm{d}t}\mathbb{E}\left[\frac{\mathrm{d}u'(c_t)}{u'(c_t)}\right] = \mathbb{E}(i^f_t + RP_t - \pi_t)$$

with

$$\begin{aligned} RP_t \;\equiv\;& -\tilde{c}_d(\mathbb{Z}_t;\mathbb{Y}_t)^2\sigma_d^2 - \tilde{c}_A(\mathbb{Z}_t;\mathbb{Y}_t)^2\sigma_A^2 - \tilde{c}_g(\mathbb{Z}_t;\mathbb{Y}_t)^2\sigma_g^2 - \tilde{c}_r(\mathbb{Z}_t;\mathbb{Y}_t)^2\sigma_m^2 + \tfrac{1}{2}\tilde{c}_{dd}(\mathbb{Z}_t;\mathbb{Y}_t)\sigma_d^2 \\ & +\tfrac{1}{2}\tilde{c}_{AA}(\mathbb{Z}_t;\mathbb{Y}_t)\sigma_A^2 + \tfrac{1}{2}\tilde{c}_{gg}(\mathbb{Z}_t;\mathbb{Y}_t)\sigma_g^2 + \tfrac{1}{2}\tilde{c}_{rr}(\mathbb{Z}_t;\mathbb{Y}_t)\sigma_m^2 + \tilde{c}_d(\mathbb{Z}_t;\mathbb{Y}_t)\sigma_d^2, \end{aligned} \tag{38}$$

and thus the shadow risk free rate is $i^f_t \equiv i_t - RP_t$. Our numerical results show how the risk premium is affected by the different state variables (cf. Figure C.52).

Hence, the Euler equation of the conditional deterministic system can be written as

$$\mathrm{d}c_t = -(\rho - i_t + \pi_t + RP_t + \rho_d \log d_t - \tfrac{1}{2}\sigma_d^2)c_t\mathrm{d}t \tag{39}$$

or

$$\mathrm{d}c_t \equiv -(r_t - i_t + \pi_t)c_t\mathrm{d}t \tag{40}$$

with $r_t \equiv \rho + RP_t + \rho_d \log d_t - \frac{1}{2}\sigma_d^2$, loosely interpreted as the 'natural rate of interest' (among others Werning, 2012). This link seems important for our analysis, because it sheds light on different sources of shocks to the natural rate (temporary or permanent). Indeed, a (positive) shock d_t will increase the natural rate temporarily, while a permanent increase in the risk premium RP_t (e.g., higher fiscal policy shocks σ_g) would increase the long-end of the yield curve similar to a change of the inflation target.

3.8. Can we rule out multiple equilibria? Different answers?

In this section we study local determinacy of the full NK model. We illustrate how the results depend on the inflation target $\pi_{ss} > 0$, and how the Taylor rule can be extended to allow for larger regions of determinacy. For comparison with the simple model we assume throughout the section $s_g = 0$ and $\|(\sigma_d, \sigma_A, \sigma_g, \sigma_m)\| = 0$, such that $r_t = \rho$.

While the simple NK model with a *feedback rule* has no state variables, the NK model with no shocks (henceforth minimal NK model) with $\pi_{ss} > 0$ introduces price dispersion v_t as a relevant state variable, and a unique locally bounded solution requires three positive eigenvalues of the Jacobian matrix (cf. Appendix A.8.1)[22]

$$A_1 = \begin{bmatrix} \phi_y & 0 & (1-\phi_\pi)a_2 y_{ss}/x_{1,ss} & (\phi_\pi - 1)a_2 y_{ss}/x_{2,ss} \\ 0 & \varepsilon\pi_{ss} - \delta & -\varepsilon\pi_{ss} v_{ss}/x_{1,ss} & \varepsilon\pi_{ss} v_{ss}/x_{2,ss} \\ 0 & 0 & \rho + \varepsilon a_2 & -(\varepsilon - 1)a_2 x_{1,ss}/x_{2,ss} \\ -(1+\vartheta)a_1 x_{2,ss}/y_{ss} & -\vartheta a_1 x_{2,ss}/v_{ss} & \varepsilon a_2 x_{2,ss}/x_{1,ss} & a_1 - \varepsilon a_2 \end{bmatrix}$$

where

$$a_1 \equiv \rho + \delta - \varepsilon\pi_{ss}, \quad a_2 \equiv \delta + (1-\varepsilon)\pi_{ss}, \tag{41}$$

such that the (linearized) inflation dynamics are

$$\begin{aligned} \mathrm{d}\pi_t &= \rho(\pi_t - \pi_{ss})\,\mathrm{d}t - (\delta + (1-\varepsilon)\pi_{ss})\pi_{ss}(x_{2,t}/x_{2,ss} - 1)\mathrm{d}t \\ &\quad -\kappa((y_t/y_{ss} - 1) + (v_t/v_{ss} - 1)\vartheta/(1+\vartheta))\mathrm{d}t. \end{aligned} \tag{42}$$

So we define

$$\kappa \equiv (\delta + (1-\varepsilon)\pi_{ss})(1+\vartheta)\left(\rho + \delta - \varepsilon\pi_{ss}\right). \tag{43}$$

For a unique locally bounded equilibrium we need three positive and one negative eigenvalue. The determinacy regions are shown in the accompanying web appendix.

Similarly, in the minimal NK model with *partial adjustment*, the two relevant state variables are the interest rate and the level of price dispersion, so a unique locally bounded solution requires three positive eigenvalues of the Jacobian matrix (cf. Appendix A.8.2)

$$A_2 = \begin{bmatrix} 0 & 0 & a_2 y_{ss}/x_{1,ss} & -a_2 y_{ss}/x_{2,ss} & y_{ss} \\ 0 & \varepsilon\pi_{ss} - \delta & -\varepsilon\pi_{ss} v_{ss}/x_{1,ss} & \varepsilon\pi_{ss} v_{ss}/x_{2,ss} & 0 \\ 0 & 0 & \rho + \varepsilon a_2 & (1-\varepsilon)a_2 x_{1,ss}/x_{2,ss} & 0 \\ -(1+\vartheta)a_1 x_{2,ss}/y_{ss} & -\vartheta a_1 x_{2,ss}/v_{ss} & \varepsilon a_2 x_{2,ss}/x_{1,ss} & a_1 - \varepsilon a_2 & 0 \\ \theta\phi_y/y_{ss} & 0 & -\theta\phi_\pi a_2/x_{1,ss} & \theta\phi_\pi a_2/x_{2,ss} & -\theta \end{bmatrix}$$

whereas for $\pi_{ss} = 0$ it collapses to the 3×3 matrix of the simple model. Note that the (linearized) inflation dynamics are not affected by the specification of the Taylor rule.

For a unique locally bounded equilibrium we need three positive and two negative eigenvalues. The determinacy regions are shown in the accompanying web appendix.

[22] We impose the parametric restriction $\delta > \varepsilon\pi_{ss}$ to ensure non-negative price dispersion, which in the frictionless case $\delta \to \infty$ the condition is fulfilled. For $\pi_{ss} = 0$ the system can be reduced to

$$A_1 = \begin{bmatrix} \phi_y & \phi_\pi - 1 \\ -\kappa & \rho \end{bmatrix},$$

which shows that the output response would not introduce different conclusions regarding stability in the simple NK model: A necessary (and sufficient) condition for local determinacy still would be $\phi_\pi > 1$.

Our results highlight potential pitfalls when letting the inflation target vary as an random walk (e.g., Ireland, 2007), because of the inability of the model to nail down unexpected inflation even with active monetary policy ($\phi_\pi > 1$). This is in sharp contrast to the approximation around zero inflation targets, hence the traditional requirement for active monetary policy is no longer sufficient.[23] Moreover, we know that determinacy regions are quite sensitive to the timing assumption (Hintermaier, 2005): For the continuous-time limit, a positive output response, $\phi_y > 0$ helps in the feedback model. To summarize, changing the inflation target is likely to have completely different implication than the nonlinear model (or its linearized version), compared to the linearized model around zero inflation target, because of the solution may exhibit multiple (sunspot) equilibria.

3.9. Which policy instruments? Different answers?

The nonlinear approach also sheds new light on the policy instruments, because our results are pointing towards the fact that the variation in the (perception of the) inflation target can be triggered by changes in risk premia, which may reflect changes in future uncertainty. Hanson and Stein (2015) find that (the more narrow interpretation of) forward guidance in the NK model (excluding shocks to target rates) is not able to generate the observed effects in the real yield curve. They argue that changes in term premia are more likely, but which are absent in the linear (simple) NK model. Though quantitatively those effects are *not* important in the presented parsimonious model, it gives directions for developing the theory and provides one alternative explanation for changes at the longer end of the yield curve through the risk channel beyond the potential policy instrument π_{ss}.

Apart from the effects of risk, the policy instruments are the same as before. The more general Taylor rules (20a) and (20b) introduce an output response ϕ_y, in addition to the inflation response ϕ_π as a new policy parameter.

3.10. Do higher interest rates raise or lower inflation? Different answers?

Let us consider unique locally bounded equilibria for the remaining analysis.[24] We study the link between inflation and (short-term) interest rates in the full (nonlinear) NK model and compare it to the insights from the simple NK model. We also study the impulse response function from the nonlinear approach for two identical economies, one of them is initialized at the steady state values and the other one for interest rates near zero. Does the same policy experiment imply the same predictions for both scenarios?

[insert Figure 4]

[23]Note that this result is not an artefact of the continuous-time approach. A similar result regarding determinacy is obtained in the discrete-time model (Coibion and Gorodnichenko, 2011).

[24]As we enter the indeterminacy region, the sharp prediction of the simple NK model is lost. While the inflation target for the simple model does not matter for determinacy, it is important for the dynamics of the full model and the two models would have completely different predictions (see Section 3.8).

Similar to the simple NK model, for the partial adjustment model (and given π_{ss}), the inflation rate is a negative function of the nominal interest rate (see Figure 4). This effect, however, is much smaller than suggested by the simple model. For our parameterization, inflation falls only by 0.1 percentage points on impact for an 1 percentage point increase of interest rates, around the steady state (slightly smaller around near-zero interest rates). We find that the nonlinear model and the linearized version (with non-zero targets) give a similar picture, so we conclude that the counterfactual large response of inflation indeed is an artefact of linearizing around a zero inflation target rate. Although output is associated with higher inflation rates (see Phillips curve), which implies that the central bank tends to increase interest rates in times with high inflation rates, the negative link between inflation and interest rates in the NK model is *not* overturned for an output response. In contrast, for the feedback rule, we would observe a positive link between interest rates and inflation along the dimension of the relevant state variables (price dispersion and shocks), as long as the central bank actively follows the interest rate peg.

Let us now reconsider our thought experiment of lowering the inflation target by 50 bp, and at the same time the short-term interest rate is *decreased* by 250 bp. On impact, the inflation rate seems *unaffected* (moderately *increases* by 50 bp in the linearized version), and both inflation and interest rates accommodate their new lower equilibria after about 15 quarters. We now compare this finding to a situation, when interest rates essentially hit zero values (other state variables are at steady-state values). But also in this scenario, on impact the inflation rate would remain unaffected, and then both inflation and interest rates accommodate to lower levels. While on impact the Fisher effect (more than) offsets the traditional negative trade-off, in the long run inflation and interest rates are expected to decline to their new steady-state levels (cf. Figures C.1 and C.2).

Summarizing, the choice of the Taylor rule in the (continuous-time) NK model can be decisive for the answer whether higher interest rates raise or (temporarily) lower inflation. While the feedback rule postulates that higher interest rates necessarily correspond to higher inflation rates (varying the relevant state variables/shocks), the partial adjustment model supports both a negative and a positive link as in the simple model. Our results indicate that the policy experiments imply qualitatively the same responses for interest rates at near zero values compared to normal times about the long-run equilibrium.

3.11. Can we explain the recent episodes? Different answers?

For comparison with the simple model, we abstract from the effects of uncertainty in this section. Although the source of the target shocks may in fact be unexpected changes in the degree of uncertainty, the parsimonious model presented here does not generate sufficient risk premia in order to overturn the results below. Thus, without loss of generality we compare the effects of the simple model vs. the nonlinear approach. We discuss the effects of uncertainty and how it affects risk premia thereafter.

In what follows we show the insights for the ability of the full *nonlinear* NK model to explain the recent episodes: (i) with sign reversal (2001-2007), (ii) including the zero-interest-rate policy (2007-2015), (iii) with inflation rebound *and* near-zero interest rates (2011), and (iv) including an apparent term structure anomaly (2004-2005).

[insert Table 4]

3.11.1. Sign reversal revisited

Comparing the simulated monetary policy shock of 500 bp (and target shock of 200 bp) to the results of the simple model we find that the problematic prediction of the NK model for the period 2001-2007, e.g., the counterfactual increase of the inflation rate by 250 bp, is in fact an artefact of the linearized model around zero inflation target. Note that this result is *not* a problem of linearization per se, but rather due to neglecting the non-zero inflation target (and price dispersion).[25] The full nonlinear approach with both temporary and permanent shock to monetary policy predicts a similar $\vee$ pattern of Fed Funds, 10Y yields and core CPI inflation (cf. Table 4, Figures C.5 and C.11).

The linearized model around non-zero inflation targets shows quite similar dynamics compared to the nonlinear model. The predictions of the simple model, however, are useful though to look at the long-run properties of the NK model. The different predictions are only relevant for the short-run dynamics, as the models inherit the same fixed point.

3.11.2. ZIRP period revisited

Let us now consider the monetary policy shocks together with a preference shock. The new insights we get are really due to the nonlinear model, and *not* only an artefact of neglecting non-zero inflation targets in the simple model. We simulate the monetary policy shock of 475 bp (and target shock of 200 bp) together with a 'preference shock' of about 10 percent, which is assumed to follow the logistic process (cf. Appendix A.9):

$$\mathrm{d}d_t = \rho_d(d_t - \bar{d})\,(1 - d_t)\,/(1 - \bar{d})\,\mathrm{d}t, \quad d_t > \bar{d}, \tag{44}$$

with $\bar{d} = 0.9130$ and $\rho_d = 0.975$. It implies that the initial value $d_0 = 0.9220$ is 1 percent above the lower bound.[26] In other words, this shock is considered 'large' and thus will have completely different dynamics than small shocks. This particular parameterization has been chosen to show that the implied interest rate process (of the full nonlinear approach)

[25] The linearized model around 2 percent inflation target without a target shock now predicts only a small increase by 50 bp. But still, the same puzzling 'sign reversal' emerges qualitatively.

[26] Note that with the assumed logistic process for the preference shock the Euler equation (32) and the definition of the risk premium (38) changes for the nonlinear model (cf. Section A.2).

now is a prolonged period of an *apparently* binding ZLB (cf. Table 4, Figures 6 and 7). This implies a ZIRP period of about 5 quarters, and is consistent with $\phi > 1$.

[insert Figures 6 and 7]

Note that the linearized model around non-zero inflation targets shows quite similar dynamics as the simple NK model, but strikingly fails to capture the nonlinear effects of the logistic process (cf. Figure C.25). This result is quite intuitive because the dynamics of the linear model (5) are the same as (44) only for small shocks.

One may ask why researches did not come across imposing different processes before? A potential explanation is that in 'normal times' with smaller shocks, the local dynamics of the assumed process (5) could have been quite successful. In normal times, when the ZLB was out of reach, the (unobserved) shocks might have been well described by the simple OU process. In fact, the local dynamics of the logistic process are observationally equivalent to the dynamics of the OU process (cf. Appendix A.9). Thus, the implications for the model dynamics can be different in non-normal times. Moreover, we show that such shocks must be large in order to drag the interest rate close to (potentially below) zero values. Hence, the traditional (linear and local) approach is not appropriate for large shocks. This confirms the result in Brunnermeier and Sannikov (2014) such that accounting for nonlinearities is an important issue in times of crises.

So distinguishing between normal times and non-normal times, in which the dynamics are different from those at the intended equilibrium point, is one alternative interpretation in which a single preference shock generates the observed pattern in the interest rates. We conclude that (44) is a parsimonious specification where the dynamics of large negative shocks (non-normal times) are different from the dynamics of a small shock (normal times). At the same time, the linear approximation of (44) replicates the OU process (5).

3.11.3. Inflation rebound (near-zero interest rates) revisited

Recall that the simple NK model fails to replicate the observed pattern in the data even if a target shock of 200 bp is accompanied by a negative preference shock of about 15 percent. We now simulate the same 'shock' to the natural rate as in the simple NK model, which is assumed to follow (44) with $\bar{d} = 0.8667$ and $\rho_d = 0.975$, in the full nonlinear approach. The presumed shock implies that $d_0 = 0.8695$ is only about 0.5 percent above the lower bound $\bar{d}$, which makes the shock indeed *large* compared to normal times. This parameterization is chosen to show the ability of the NK model to predict immobile interest rates near zero and inflation to rebound for a single shock.[27] As a result, we find indeed that inflation may rebound (about 1 percent on impact), and then returns to the target rate of about 2

[27]In particular, we do not claim that this parameterization reflects the true data generating process. It remains an empirical task to estimate the structural parameters of the model including $\bar{d}$ and ρ_d.

percent, with interest rates being immobile for another 5 quarters, and eventually to lift off to higher values (cf. Table 4, Figures C.29 and C.30).

Similar to the ZIRP period, the linear model (around zero/non-zero inflation targets) fails to capture the nonlinear effects and is *not* able to generate immobile interest rates at near zero values consistent with the Taylor principle and $\phi > 1$.

3.11.4. Term structure anomalies revisited

Similar to the simple model, the rotation in the yield curve between 2004 and 2005 is obtained if the observed monetary policy shock is accompanied by a preference shock. For comparison with the simple model we set $\bar{d} = 0$ (i.e., nonlinear effects are less important) together with $\rho_d = 0.4214$. This refers to the logistic growth model with similar dynamics as the OU process nearby the steady state. We simulate the effects of a monetary policy shock increasing the interest rate by 150 bp, which is accompanied by a negative shock to the natural rate of about 10 percent (keeping the inflation target about 2 percent).

Similar to the simple NK model, the simulated response basically replicates dynamics of macro aggregates and the rotation in both yield curves (cf. Figures C.35 and C.36). In fact, the predicted inflation dynamics are more realistic, on impact it drops by 50 bp and then increases back to the target level (cf. Table 4).[28] Our findings are consistent with the 'expectation hypothesis' explanation in the literature, which says that the term structure simply reflects expectations of future inflation and the output gap. Hence, if the FOMC raises policy rates today but, because of lower expected inflation, this leads agents to anticipate lower short-term interest rates in the future, then long-term interest rates could actually decrease (cf. Gürkaynak and Wright, 2012, p.333).

The linearized model around non-zero inflation targets shows quite similar dynamics compared to the nonlinear model. So apart from the alternative specification for the preference shock dynamics, the parsimonious NK model presented in the main text does not inherit important nonlinearities. Although a preference shock of the same order of magnitude as during the ZIRP period is assumed, here the initial value for d_0 is sufficiently above its 'natural' bound $\bar{d} = 0$, so the nonlinear dynamics do not matter here.

3.12. Discussion of the new insights

The full (nonlinear) approach and the local dynamics around positive inflation targets give rise to at least four insights. First, the effects of risk affect the long-term interest rates and are one potential explanation for changes at the long-end of the yield curve (term premia) similar to changes in the inflation target. This risk channel, however, is negligible in the standard NK model, but can be relevant in models where risk matters quantitatively

[28] We abstract from 'adjusting' price dispersion downwards in 2004 for our experiment, which in fact would decrease the level of inflation. Hence, the predicted rate is even interpreted as an upper bound.

for asset pricing (see Parra-Alvarez, Polattimur, and Posch, 2018).[29] Second, shifts in monetary policy, or its perception, are important to generate the inflation dynamics and the correlation with interest rates as in the data. We confirm the hypothesis that inflation persistence is driven by variations in long-run policy (cf. Cogley and Sbordone, 2008). Third, we replicate the findings in Coibion and Gorodnichenko (2011), showing that the conclusion about determinacy in the NK model is different in models with positive trend inflation. This is important, because the numerical solution approach can determine the theoretical prediction of the model. Similarly we find that the output response helps to obtain determinacy in the feedback model, whereas the partial adjustment model seems to be more robust to positive inflation target because of the interest smoothing component. Fourth, a nonlinear approach can generate a ZIRP period in which $\phi > 1$.

Our results document the ability of the NK model to explain the recent episodes, including the ZIRP episode with active monetary policy, $\phi > 1$, which is in line with the theoretical prediction at both ends of the yield curve (cf. Table 4). Hence, we contribute to the recent discussion because the standard approach is *not* able to explain a prolonged period of stable quiet inflation at near-zero interest rates. We show that the nonlinear NK model is able to generate phenomena similar to those observed in the data, e.g., by allowing for a logistic shock process to the natural rate, which is observational equivalent to the standard OU process at the long-run equilibrium values, but has quite different dynamics for large shocks. We also show that the implied yield curve motivates and supports changing target rates as suggested in the empirical literature.

We do not advocate that the ZIRP period has been generated by a preference shock. Among others, a persistent monetary policy shock may generate similar dynamics. It is well known that identification of shocks is not an easy task, but this is a regular task of empirical macroeconomics, so it remains an empirical question and "not a special task that must be relegated to theory or philosophy alone" (Cochrane, 2017b, p.61). We show that the nonlinear approach largely extends the theoretical model predictions and suggest a potential remedy to the standard NK model. Our experiments, in particular those for unconventional monetary policy, challenge some of the standard identification schemes, e.g., those restricting monetary policy instruments not to affect long-run economic activity (for recent empirical identification schemes, see Ramey, 2016; Uribe, 2017).

4. Conclusion

In this paper we show the ability of the NK model to explain the recent episodes, even when restricting ourselves to the regions of (local) determinacy. We show that the NK model with active monetary policy supports both views that higher interest rates result

[29]Introducing recursive preferences produces a large and variable term premium without compromising the model's ability to fit key macroeconomic variables (cf. Rudebusch and Swanson, 2012).

into higher long-run inflation (neo-Fisherian view), but at the same time higher interest can temporarily reduce inflation (traditional view). We also show that the Taylor rule requires an interest smoothing component to obtain the temporary negative effect. Central to our finding is that monetary policy actions (changes in short-term rates) may trigger variations in long-term target rates, a view that is motivated by empirical data. Allowing further for temporary shocks to the natural rate allows us to understand several puzzles in the literature, including apparent term structure anomalies. We also show that a nonlinear approach can be used to generate a ZIRP period with stable and quiet inflation, fully consistent with the model predictions. We review several important insights related to positive trend inflation and determinacy in the NK model.

So the full NK model with positive trend inflation and the way we interpret monetary policy shocks suggests a partial remedy compared to the simple NK model, which typically is approximated around zero target rates (no trend inflation). With a nonlinear solution method, accounting for nonlinearities which can be important after 'large' shocks, the NK model may also generate a ZIRP period, in which near-zero rates are regarded as a policy choice without a threat of high inflation. We also show how uncertainty is introduced in this framework and shed light on the channels through which they affect the results.

We believe that this paper is a starting point for several lines of research. First, we may extend our approach to a medium-scale model, allowing for other nominal and/or real frictions, habit formation, variable capacity utilization and adjustment cost as in models used by central banks for policy analysis (e.g., Christiano, Eichenbaum, and Rebelo, 2011), or by including a financial sector (e.g., Brunnermeier and Sannikov, 2014). Second, we should estimate the structural parameters using empirical data. One key advantage of the continuous-time approach is that the model solution is consistent with different frequencies of macro and financial data (cf. Christensen, Posch, and van der Wel, 2016). A promising starting point is to combine traditional estimation approaches such as the particle filter with an Euler discretization scheme of the equilibrium dynamics and use the standard econometric toolbox. Alternatively, we can use the continuous-time econometric toolbox developed in the financial literature and apply them to our macro models. Third, we may study the monetary policy transmission in a heterogeneous-agent economy, e.g., with idiosyncratic income shocks (cf. Kaplan, Moll, and Violante, 2018).

Our ultimate goal is not to advocate the use of continuous time over discrete time in all applications. In this paper, we merely show how to handle models in macroeconomics which exhibit nonlinearities, and where a local approximation scheme might fail to give the right answers. Although formulating and solving complicated problems in macroeconomics is important, we also aim to provide a simple device to develop intuition, conceptualize, and facilitate the way we think about dynamic problems in economics.

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A. Appendix

A.1. Technical details

 # $\mathrm{d}a_t$ on p.16: The household can trade on Arrow securities (excluded to save on notation) and on a nominal government bonds b_t at a nominal interest rate of i_t. Let n_t denote the number of shares and p_t^b the equilibrium price of bonds. Suppose the household earns a disposable income of $i_t b_t + p_t w_t l_t + p_t T_t + p_t \digamma_t$, where p_t is the price level (or price of the consumption good), w_t is the real wage, T_t is a lump-sum transfer, and $\digamma_t$ are the profits of the firms in the economy; the household's budget constraint is:

$$\mathrm{d}n_t = \frac{i_t b_t - p_t c_t + p_t w_t l_t + p_t T_t + p_t \digamma_t}{p_t^b}\mathrm{d}t. \tag{A.1}$$

Let bond prices follow:

$$\mathrm{d}p_t^b = \alpha_t p_t^b \mathrm{d}t \tag{A.2}$$

in which α_t denotes a price change, which is determined in general equilibrium (in equilibrium prices are function of the state variables, for example, by fixing α_t the bond supply has to accommodate so as to permit the bond's nominal interest rate being admissible). The household's financial wealth, $b_t = n_t p_t^b$, is then given by:

$$\mathrm{d}b_t = (i_t b_t - p_t c_t + p_t w_t l_t + p_t T_t + p_t \digamma_t)\mathrm{d}t + \alpha_t b_t \mathrm{d}t, \tag{A.3}$$

Let prices p_t follow the process:

$$\mathrm{d}p_t = \pi_t p_t \mathrm{d}t \tag{A.4}$$

such that the (realized) rate of inflation is locally non-stochastic. We can interpret $\mathrm{d}p_t/p_t$ as the realized inflation over the period $[t, t+\mathrm{d}t]$ and π_t as the inflation rate.

Letting $a_t \equiv b_t/p_t$ denote real financial wealth and using Itô's formula, the household's real wealth evolves according to:

$$\mathrm{d}a_t = \frac{\mathrm{d}b_t}{p_t} - \frac{b_t}{p_t^2}\mathrm{d}p_t = \frac{i_t b_t - p_t c_t + p_t w_t l_t + p_t T_t + p_t \digamma_t + \alpha_t b_t}{p_t}\mathrm{d}t - \frac{b_t}{p_t^2}\pi_t p_t \mathrm{d}t$$

or:

$$\mathrm{d}a_t = ((i_t + \alpha_t - \pi_t)a_t - c_t + w_t l_t + T_t + \digamma_t)\,\mathrm{d}t \tag{A.5}$$

Since government bonds are in net zero supply, $b_t = 0$, it implies $\alpha_t = 0$ for all t. ■

$\#\mathrm{d}x_{1,t}$ *on p.18:* Differentiating $x_{1,t}$ in (14) with respect to time gives:

$$\begin{aligned}
\frac{1}{\mathrm{d}t}\mathrm{d}x_{1,t} &= e^{(\rho+\delta)t}p_t^{1-\varepsilon}\frac{1}{\mathrm{d}t}\mathrm{d}\mathbb{E}_t\int_t^\infty \lambda_\tau e^{-(\rho+\delta)\tau}\left(\frac{1}{p_\tau}\right)^{1-\varepsilon}y_\tau \mathrm{d}\tau \\
&\quad +\mathbb{E}_t\int_t^\infty \lambda_\tau e^{-(\rho+\delta)\tau}\left(\frac{1}{p_\tau}\right)^{1-\varepsilon}y_\tau \mathrm{d}\tau\frac{1}{\mathrm{d}t}\mathrm{d}\left(e^{(\rho+\delta)t}p_t^{1-\varepsilon}\right) \\
&= -\lambda_t y_t + (\rho+\delta+(1-\varepsilon)\pi_t)\,\mathbb{E}_t\int_t^\infty \lambda_\tau e^{-(\rho+\delta)(\tau-t)}\left(\frac{p_t}{p_\tau}\right)^{1-\varepsilon}y_\tau \mathrm{d}\tau
\end{aligned}$$

or (16) in the main text. A similar procedure gives (17). ■

$\#\mathrm{d}\pi_t$ *on p.19:* Differentiating (18), we obtain the inflation dynamics as:

$$\begin{aligned}
\frac{1}{\mathrm{d}t}\mathrm{d}\pi_t &= \delta\left(\Pi_t^*\right)^{-\varepsilon}\frac{1}{\mathrm{d}t}\mathrm{d}\Pi_t^* = \delta\left(\Pi_t^*\right)^{-\varepsilon}\frac{\varepsilon}{\varepsilon-1}\frac{1}{\mathrm{d}t}\mathrm{d}\left(\frac{x_{2,t}}{x_{1,t}}\right) \\
&= \delta\left(\Pi_t^*\right)^{-\varepsilon}\frac{\varepsilon}{\varepsilon-1}\frac{1}{x_{1,t}}\left(\frac{1}{\mathrm{d}t}\mathrm{d}x_{2,t}-\frac{x_{2,t}}{x_{1,t}}\frac{1}{\mathrm{d}t}\mathrm{d}x_{1,t}\right) \\
&= \delta\left(\Pi_t^*\right)^{1-\varepsilon}\frac{1}{x_{2,t}}\left(\frac{1}{\mathrm{d}t}\mathrm{d}x_{2,t}-\frac{x_{2,t}}{x_{1,t}}\frac{1}{\mathrm{d}t}\mathrm{d}x_{1,t}\right) \\
&= \delta\left(\Pi_t^*\right)^{1-\varepsilon}\left(\frac{1}{x_{2,t}}\frac{1}{\mathrm{d}t}\mathrm{d}x_{2,t}-\frac{1}{x_{1,t}}\frac{1}{\mathrm{d}t}\mathrm{d}x_{1,t}\right) \\
&= \delta\left(\Pi_t^*\right)^{1-\varepsilon}\left(\frac{\left(\left(\delta-\varepsilon\pi_t\right)x_{2,t}-\lambda_t mc_t y_t\right)}{x_{2,t}}-\frac{\left(\left(\delta+(1-\varepsilon)\,\pi_t\right)x_{1,t}-\lambda_t y_t\right)}{x_{1,t}}\right), \\
&= -\delta\left(\Pi_t^*\right)^{1-\varepsilon}\left(\pi_t+\left(\frac{mc_t}{x_{2,t}}-\frac{1}{x_{1,t}}\right)\lambda_t y_t\right)
\end{aligned}$$

which is (19) in the main text. ■

$\#\mathrm{d}v_t$ *on p.20:* Differentiating (23) with respect to time gives:

$$\begin{aligned}
\frac{1}{\mathrm{d}t}\mathrm{d}v_t &= \delta\left(\Pi_t^*\right)^{-\varepsilon}+\int_{-\infty}^t \delta\frac{1}{\mathrm{d}t}\mathrm{d}e^{-\delta(t-\tau)}\left(\frac{p_{i\tau}}{p_t}\right)^{-\varepsilon}\mathrm{d}\tau \\
&= \delta\left(\Pi_t^*\right)^{-\varepsilon}-\delta\int_{-\infty}^t \delta e^{-\delta(t-\tau)}\left(\frac{p_{i\tau}}{p_t}\right)^{-\varepsilon}\mathrm{d}\tau+\int_{-\infty}^t \delta e^{-\delta(t-\tau)}p_{i\tau}^{-\varepsilon}\frac{1}{\mathrm{d}t}\mathrm{d}p_t^{\varepsilon}\mathrm{d}\tau \\
&= \delta\left(\Pi_t^*\right)^{-\varepsilon}-\delta v_t+\int_{-\infty}^t \delta e^{-\delta(t-\tau)}p_{i\tau}^{-\varepsilon}\varepsilon p_t^{\varepsilon-1}\frac{1}{\mathrm{d}t}\mathrm{d}p_t\mathrm{d}\tau \\
&= \delta\left(\Pi_t^*\right)^{-\varepsilon}+(\varepsilon\pi_t-\delta)v_t,
\end{aligned}$$

or (24) in the main text. ■

$\#\digamma_t$ on p.20: For aggregate profits, we use the demand of intermediate producers in (22):

$$
\begin{aligned}
\digamma_t &= \int_0^1 \left(\frac{p_{it}}{p_t} - mc_t\right) y_{it}\mathrm{d}i \\
&= y_t \int_0^1 \left(\frac{p_{it}}{p_t} - mc_t\right) \left(\frac{p_{it}}{p_t}\right)^{-\varepsilon} \mathrm{d}i \\
&= \left(\int_0^1 \left(\frac{p_{it}}{p_t}\right)^{1-\varepsilon} \mathrm{d}i - mc_t v_t\right) y_t \\
&= (1 - mc_t v_t) y_t
\end{aligned}
$$

which is (25) in the main text. ■

$\#V(\mathbb{Z}_t, \mathbb{X}_t)$ on p.21: The HJB equation (26) in scalar notation reads

$$
\begin{aligned}
\rho V(\mathbb{Z}_t; \mathbb{Y}_t) &= \max_{(c_t, l_t)} d_t \left\{\log c_t - \psi \frac{l_t^{1+\vartheta}}{1+\vartheta}\right\} \\
&\quad + ((i_t - \pi_t)a_t - c_t + w_t l_t + T_t + \digamma_t) V_a \\
&\quad + (\theta\phi_\pi(\pi_t - \pi_{ss}) + \theta\phi_y(y_t/y_{ss} - 1) - \theta(i_t - i_{ss}))V_r + \tfrac{1}{2}\sigma_m^2 V_{rr} \\
&\quad + \left(\delta\,(\Pi_t^*)^{-\varepsilon} + (\varepsilon\pi_t - \delta)v_t\right) V_v \\
&\quad - (\rho_d \log d_t - \tfrac{1}{2}\sigma_d^2) d_t V_d + \tfrac{1}{2}\sigma_d^2 d_t^2 V_{dd} \\
&\quad - (\rho_A \log A_t - \tfrac{1}{2}\sigma_A^2) A_t V_A + \tfrac{1}{2}\sigma_A^2 A_t^2 V_{AA} \\
&\quad - (\rho_g \log s_{g,t} - \tfrac{1}{2}\sigma_g^2) s_{g,t} V_g + \tfrac{1}{2}\sigma_g^2 s_{g,t}^2 V_{gg}. \qquad \text{(A.6)}
\end{aligned}
$$

■

$\# dV_a(\mathbb{Z}_t, \mathbb{X}_t)$ on p.22: From A.6, the concentrated HJB equation in scalar notation reads

$$
\begin{aligned}
\rho V(\mathbb{Z}_t; \mathbb{Y}_t) &= d_t \log c(\mathbb{Z}_t; \mathbb{Y}_t) - d_t \psi \frac{l(\mathbb{Z}_t; \mathbb{Y}_t)^{1+\vartheta}}{1+\vartheta} \\
&\quad + ((i_t - \pi_t)a_t - c(\mathbb{Z}_t; \mathbb{Y}_t) + w_t l(\mathbb{Z}_t; \mathbb{Y}_t) + T_t + \digamma_t) V_a \\
&\quad + (\theta\phi_\pi(\pi_t - \pi_{ss}) + \theta\phi_y(y_t/y_{ss} - 1) - \theta(i_t - i_{ss}))V_r + \tfrac{1}{2}\sigma_m^2 V_{rr} \\
&\quad + \left(\delta\,(\Pi_t^*)^{-\varepsilon} + (\varepsilon\pi_t - \delta)v_t\right) V_v \\
&\quad - (\rho_d \log d_t - \tfrac{1}{2}\sigma_d^2) d_t V_d + \tfrac{1}{2}\sigma_d^2 d_t^2 V_{dd} \\
&\quad - (\rho_A \log A_t - \tfrac{1}{2}\sigma_A^2) A_t V_A + \tfrac{1}{2}\sigma_A^2 A_t^2 V_{AA} \\
&\quad - (\rho_g \log s_{g,t} - \tfrac{1}{2}\sigma_g^2) s_{g,t} V_g + \tfrac{1}{2}\sigma_g^2 s_{g,t}^2 V_{gg}. \qquad \text{(A.7)}
\end{aligned}
$$

Using the envelope theorem, we obtain the costate variable V_a as:

$$\begin{aligned}
\rho V_a \quad = \quad & (i_t - \pi_t)V_a + ((i_t - \pi_t)a_t - c_t + w_t l_t + T_t + F_t)\, V_{aa} \\
& + (\theta\phi_\pi(\pi_t - \pi_{ss}) + \theta\phi_y(y_t/y_{ss} - 1) - \theta(i_t - i_{ss}))V_{ra} + \tfrac{1}{2}\sigma_m^2 V_{rra} \\
& + \left(\delta\left(\Pi_t^*\right)^{-\varepsilon} + (\varepsilon\pi_t - \delta)v_t\right) V_{va} \\
& - (\rho_d \log d_t - \tfrac{1}{2}\sigma_d^2)d_t V_{da} + \tfrac{1}{2}\sigma_d^2 d_t^2 V_{dda} \\
& - (\rho_A \log A_t - \tfrac{1}{2}\sigma_A^2)A_t V_{Aa} + \tfrac{1}{2}\sigma_A^2 A_t^2 V_{AAa} \\
& - (\rho_g \log s_{g,t} - \tfrac{1}{2}\sigma_g^2)s_{g,t} V_{ga} + \tfrac{1}{2}\sigma_g^2 s_{g,t}^2 V_{gga}.
\end{aligned} \tag{A.8}$$

An alternative formulation in terms of differentials is:

$$\begin{aligned}
(\rho - i_t + \pi_t)\, V_a \mathrm{d}t \;=\; & V_{aa}\mathrm{d}a_t + (\mathrm{d}i_t - \sigma_m \mathrm{d}B_{m,t})V_{ra} + \tfrac{1}{2}\sigma_m^2 V_{rra} + V_{va}\mathrm{d}v_t \\
& + (\mathrm{d}d_t - \sigma_d d_t \mathrm{d}B_{d,t})\, V_{da} + \tfrac{1}{2}\sigma_d^2 d_t^2 V_{dda}\mathrm{d}t \\
& + (\mathrm{d}A_t - \sigma_A A_t \mathrm{d}B_{A,t})\, V_{Aa} + \tfrac{1}{2}\sigma_A^2 A_t^2 V_{AAa}\mathrm{d}t + (\mathrm{d}s_{g,t} - \sigma_g s_{g,t}\mathrm{d}B_{g,t})\, V_{ga} + \tfrac{1}{2}\sigma_g^2 s_{g,t}^2 V_{gga}\mathrm{d}t
\end{aligned}$$

or

$$\begin{aligned}
& (\rho - i_t + \pi_t)\, V_a \mathrm{d}t + \sigma_d d_t V_{da}\mathrm{d}B_{d,t} + \sigma_A A_t V_{Aa}\mathrm{d}B_{A,t} + \sigma_g s_{g,t} V_{ga}\mathrm{d}B_{g,t} + \sigma_m i_t V_{ra}\mathrm{d}B_{m,t} \\
= \quad & V_{aa}\mathrm{d}a_t + V_{ra}\mathrm{d}i_t + \tfrac{1}{2}\sigma_m^2 i_t^2 V_{rra} + V_{va}\mathrm{d}v_t \\
& + V_{da}\mathrm{d}d_t + \tfrac{1}{2}\sigma_d^2 d_t^2 V_{da}\mathrm{d}t + V_{Aa}\mathrm{d}A_t + \tfrac{1}{2}\sigma_A^2 A_t^2 V_{Aa}\mathrm{d}t + V_{ga}\mathrm{d}s_{g,t} + \tfrac{1}{2}\sigma_g^2 s_{g,t}^2 V_{ga}\mathrm{d}t.
\end{aligned}$$

Observe that the costate variable in general evolves according to:

$$\begin{aligned}
\mathrm{d}V_a \quad = \quad & V_{aa}\mathrm{d}a_t + V_{ra}\mathrm{d}i_t + \tfrac{1}{2}\sigma_m^2 V_{rra}\mathrm{d}t + V_{va}\mathrm{d}v_t \\
& + V_{da}\mathrm{d}d_t + \tfrac{1}{2}\sigma_d^2 d_t^2 V_{dda}\mathrm{d}t + V_{Aa}\mathrm{d}A_t + \tfrac{1}{2}\sigma_A^2 A_t^2 V_{AAa}\mathrm{d}t + V_{ga}\mathrm{d}s_{gt} + \tfrac{1}{2}\sigma_g^2 s_{g,t}^2 V_{gga}\mathrm{d}t \\
= \quad & (\rho - i_t + \pi_t)V_a\mathrm{d}t \\
& + \sigma_d d_t V_{da}\mathrm{d}B_{d,t} + \sigma_A A_t V_{Aa}\mathrm{d}B_{A,t} + \sigma_g s_{g,t} V_{ga}\mathrm{d}B_{g,t} + \sigma_m V_{ra}\mathrm{d}B_{m,t},
\end{aligned}$$

which is (30) in the main text. ■

 # m_s/m_t (SDF) on p.22: Starting from (30):

$$\begin{aligned}
\mathrm{d}\ln V_a \quad = \quad & \frac{1}{V_a}\mathrm{d}V_a - \tfrac{1}{2}\sigma_d^2 d_t^2 \frac{V_{da}^2}{V_a^2}\mathrm{d}t - \tfrac{1}{2}\sigma_A^2 A_t^2 \frac{V_{Aa}^2}{V_a^2}\mathrm{d}t - \tfrac{1}{2}\sigma_g^2 s_{g,t}^2 \frac{V_{ga}^2}{V_a^2}\mathrm{d}t - \tfrac{1}{2}\sigma_m^2 \frac{V_{ra}^2}{V_a^2}\mathrm{d}t \\
= \quad & (\rho - i_t + \pi_t)\mathrm{d}t + \sigma_d d_t \frac{V_{da}}{V_a}\mathrm{d}B_{d,t} + \sigma_A A_t \frac{V_{Aa}}{V_a}\mathrm{d}B_{A,t} + \sigma_g s_{g,t}\frac{V_{ga}}{V_a}\mathrm{d}B_{g,t} \\
& + \sigma_m \frac{V_{ra}}{V_a}\mathrm{d}B_{m,t} - \tfrac{1}{2}\sigma_d^2 d_t^2 \frac{V_{da}^2}{V_a^2}\mathrm{d}t - \tfrac{1}{2}\sigma_A^2 A_t^2 \frac{V_{Aa}^2}{V_a^2}\mathrm{d}t - \tfrac{1}{2}\sigma_g^2 s_{g,t}^2 \frac{V_{ga}^2}{V_a^2}\mathrm{d}t - \tfrac{1}{2}\sigma_m^2 \frac{V_{ra}^2}{V_a^2}\mathrm{d}t.
\end{aligned}$$

For $s > t$, we may write:

$$e^{-\rho(s-t)}\frac{V_a(\mathbb{Z}_s;\mathbb{Y}_s)}{V_a(\mathbb{Z}_t;\mathbb{Y}_t)} =$$

$$\exp\left(\begin{array}{c} -\int_t^s(i_u-\pi_u)\mathrm{d}u-\frac{1}{2}\int_t^s\frac{V_{da}^2}{V_a^2}\sigma_d^2 d_u^2\mathrm{d}u-\frac{1}{2}\int_t^s\frac{V_{Aa}^2}{V_a^2}\sigma_A^2 A_u^2\mathrm{d}u \\ -\frac{1}{2}\int_t^s\frac{V_{ga}^2}{V_a^2}\sigma_g^2 s_{g,u}^2\mathrm{d}u-\frac{1}{2}\int_t^s\frac{V_{ra}^2}{V_a^2}\sigma_m^2\mathrm{d}u \\ +\int_t^s\frac{V_{da}}{V_a}\sigma_d d_u\mathrm{d}B_{d,u}+\int_t^s\frac{V_{Aa}}{V_a}\sigma_A A_u\mathrm{d}B_{A,u}+\int_t^s\frac{V_{ga}}{V_a}\sigma_g s_{g,u}\mathrm{d}B_{g,u}+\int_t^s\frac{V_{ra}}{V_a}\sigma_m\mathrm{d}B_{m,u}\end{array}\right).$$

which denotes the equilibrium SDF m_s/m_t in (31). ■

A.2. Obtaining the Euler equation

Using the first-order condition (27) and (30), we obtain the implicit Euler equation:

$$\mathrm{d}\left(\frac{d_t}{c_t}\right) = (\rho - i_t + \pi_t)\left(\frac{d_t}{c_t}\right)\mathrm{d}t$$
$$+\sigma_d d_t\left(\frac{1}{c_t}-\frac{d_t}{c_t^2}c_d\right)\mathrm{d}B_{d,t}-\sigma_A A_t\frac{d_t}{c_t^2}c_A\mathrm{d}B_{A,t}-\sigma_g s_{g,t}\frac{d_t}{c_t^2}c_g\mathrm{d}B_{g,t}-\sigma_m\frac{d_t}{c_t^2}c_r\mathrm{d}B_{m,t}.$$

$V_{ad} = -\left(d_t/c_t^2\right)c_d + 1/c_t$, $V_{Aa} = -\left(d_t/c_t^2\right)c_A$, $V_{ga} = -\left(d_t/c_t^2\right)c_g$, and $V_{ra} = -\left(d_t/c_t^2\right)c_r$ are expressed in terms of derivatives and levels of the consumption function. This equation has a simple interpretation: the change in the marginal utility of consumption depends on the rate of time preference minus the effective real interest rate and four additional terms that control for the innovations to the four shocks to the economy.

Hence, by applying Itô's formula we obtain the Euler equation:

$$\mathrm{d}\left(\frac{c_t}{d_t}\right) = -\left(\frac{d_t}{c_t}\right)^{-2}\left[(\rho - i_t + \pi_t)\left(\frac{d_t}{c_t}\right)\mathrm{d}t\right.$$
$$\left.+\sigma_d\left(\frac{d_t}{c_t}-\frac{d_t^2}{c_t^2}c_d\right)\mathrm{d}B_{d,t}-\sigma_A A_t\frac{d_t}{c_t^2}c_A\mathrm{d}B_{A,t}-\sigma_g s_{g,t}\frac{d_t}{c_t^2}c_g\mathrm{d}B_{g,t}-\sigma_m\frac{d_t}{c_t^2}c_r\mathrm{d}B_{m,t}\right]$$
$$+\left(\frac{d_t}{c_t}\right)^{-3}\left(\sigma_d^2\left(\frac{d_t^2}{c_t^2}-2\frac{d_t^3}{c_t^3}c_d+\frac{d_t^4}{c_t^4}c_d^2\right)+\sigma_A^2 A_t^2\frac{d_t^2}{c_t^4}c_A^2+\sigma_g^2 s_{g,t}^2\frac{d_t^2}{c_t^4}c_g^2+\sigma_m^2\frac{d_t^2}{c_t^4}c_r^2\right)\mathrm{d}t,$$

which simplifies to

$$\mathrm{d}\left(\frac{c_t}{d_t}\right) = -(\rho - i_t + \pi_t)\left(\frac{c_t}{d_t}\right)\mathrm{d}t$$
$$-\sigma_d\left(\frac{c_t}{d_t}-c_d\right)\mathrm{d}B_{d,t}+\sigma_A A_t d_t^{-1}c_A\mathrm{d}B_{A,t}+\sigma_g s_{g,t}d_t^{-1}c_g\mathrm{d}B_{g,t}+\sigma_m d_t^{-1}c_r\mathrm{d}B_{m,t}$$
$$+\left(\sigma_d^2\left(\frac{c_t}{d_t}-2c_d+\frac{d_t}{c_t}c_d^2\right)+\sigma_A^2 A_t^2\frac{d_t^{-1}}{c_t}c_A^2+\sigma_g^2 s_{g,t}^2\frac{d_t^{-1}}{c_t}c_g^2+\sigma_m^2\frac{d_t^{-1}}{c_t}c_r^2\right)\mathrm{d}t,$$

or

$$
\begin{aligned}
\mathrm{d}c_t &= -(\rho - i_t + \pi_t)c_t\mathrm{d}t + \sigma_d^2\frac{d_t^2}{c_t}c_d^2\mathrm{d}t + \sigma_A^2\frac{A_t^2}{c_t}c_A^2\mathrm{d}t + \sigma_g^2\frac{s_{g,t}^2}{c_t}c_g^2\mathrm{d}t + \sigma_m^2\frac{1}{c_t}c_r^2\mathrm{d}t \\
&\quad +\sigma_d c_d d_t \mathrm{d}B_{d,t} + \sigma_A A_t c_A \mathrm{d}B_{A,t} + \sigma_g s_{g,t} c_g \mathrm{d}B_{g,t} + \sigma_m c_r \mathrm{d}B_{m,t} \\
&\quad -c_t\rho_d \log d_t \mathrm{d}t + \tfrac{1}{2}c_t\sigma_d^2\mathrm{d}t - c_d d_t \sigma_d^2 \mathrm{d}t, \qquad \text{(A.9)}
\end{aligned}
$$

which is (32), and $c_t = c(\mathbb{Z}_t; \mathbb{Y}_t)$ denotes the household's consumption function. A similar approach implies the Euler equation for the alternative shock process as:

$$
\begin{aligned}
\mathrm{d}c_t &= -(\rho - i_t + \pi_t)c_t\mathrm{d}t + \sigma_A^2\frac{A_t^2}{c_t}c_A^2\mathrm{d}t + \sigma_g^2\frac{s_{g,t}^2}{c_t}c_g^2\mathrm{d}t + \sigma_m^2\frac{1}{c_t}c_r^2\mathrm{d}t \\
&\quad +\sigma_A A_t c_A \mathrm{d}B_{A,t} + \sigma_g s_{g,t} c_g \mathrm{d}B_{g,t} + \sigma_m c_r \mathrm{d}B_{m,t} \\
&\quad c_t\rho_d(d_t - \bar{d})\,(1 - d_t)\,/(1 - \bar{d})/d_t\mathrm{d}t. \qquad \text{(A.10)}
\end{aligned}
$$

A.3. Equilibrium

We define the recursive-competitive equilibrium of the nonlinear NK model with shocks by the sequence $\{\lambda_t, l_t, a_t, mc_t, x_{1,t}, x_{2,t}, \digamma_t, w_t, i_t, g_t, T_t, \pi_t, \Pi_t^*, v_t, y_t, d_t, A_t, s_{g,t}\}_{t=0}^{\infty}$, which is determined by the following equations:

- Euler equation, the first-order conditions of the household, and budget constraint:

Equation 1

$$
\begin{aligned}
\mathrm{d}c_t = -(\rho - i_t + \pi_t - \sigma_A^2\tilde{c}_A^2 - \sigma_g^2\tilde{c}_g^2 - \sigma_m^2\tilde{c}_r^2 + \rho_d \log d_t + (\tilde{c}_d(1 - \tilde{c}_d) - \tfrac{1}{2})\sigma_d^2)c_t\mathrm{d}t \\
+\sigma_d\tilde{c}_d c_t \mathrm{d}B_{d,t} + \sigma_A\tilde{c}_A c_t \mathrm{d}B_{A,t} + \sigma_g\tilde{c}_g c_t \mathrm{d}B_{g,t} + \sigma_m\tilde{c}_r c_t \mathrm{d}B_{m,t}
\end{aligned}
$$

Equation 2

$$\psi l_t^{\vartheta} c_t = w_t$$

Equation 3

$$d_t/c_t = \lambda_t$$

(redundant)

$$\mathrm{d}a_t = ((i_t - \alpha_t - \pi_t)a_t - c_t + w_t l_t + T_t + \digamma_t)\,\mathrm{d}t$$

- Profit maximization is given by:

Equation 4

$$\Pi_t^* = \frac{\varepsilon}{\varepsilon - 1} \frac{x_{2,t}}{x_{1,t}}$$

Equation 5

$$\mathrm{d}x_{1,t} = ((\rho + \delta - (\varepsilon - 1)\pi_t)x_{1,t} - \lambda_t y_t)\,\mathrm{d}t$$

Equation 6

$$\mathrm{d}x_{2,t} = ((\rho + \delta - \varepsilon\pi_t)\, x_{2,t} - \lambda_t y_t mc_t)\,\mathrm{d}t$$

Equation 7

$$\digamma_t = (1 - mc_t v_t) y_t$$

Equation 8

$$w_t = A_t mc_t$$

- Government policy:

Equation 9

$$\mathrm{d}i_t = (\theta\phi_\pi(\pi_t - \pi_{ss}) + \theta\phi_y(y_t/y_{ss} - 1) - \theta(i_t - i_{ss}))\mathrm{d}t + \sigma_m \mathrm{d}B_{m,t}$$

Equation 10

$$g_t = s_g s_{g,t} y_t$$

(redundant)

$$T_t = -i_t a_t - s_g s_{g,t} y_t$$

- Inflation evolution and price dispersion:

Equation 11

$$\pi_t = \frac{\delta}{1 - \varepsilon} \left((\Pi_t^*)^{1-\varepsilon} - 1 \right)$$

Equation 12

$$\mathrm{d}v_t = \left(\delta \left(\Pi_t^* \right)^{-\varepsilon} + (\varepsilon\pi_t - \delta) v_t \right) \mathrm{d}t$$

- Market clearing on goods and labor markets:

  Equation 13

$$y_t = c_t + g_t \quad \text{(expenditure)}$$

  Equation 14

$$y_t = \frac{A_t}{v_t} l_t \quad \text{(production)}$$

  (redundant)

$$y_t = w_t l_t + F_t \quad \text{(income)}$$

- Stochastic processes follow:

  Equation 15

$$\mathrm{d}d_t = -\left(\rho_d \log d_t - \tfrac{1}{2}\sigma_d^2\right) d_t \mathrm{d}t + \sigma_d d_t \mathrm{d}B_{d,t}$$

  Equation 16

$$\mathrm{d}A_t = -\left(\rho_A \log A_t - \tfrac{1}{2}\sigma_A^2\right) A_t \mathrm{d}t + \sigma_A A_t \mathrm{d}B_{A,t}$$

  Equation 17

$$\mathrm{d}s_{g,t} = -\left(\rho_g \log s_{g,t} - \tfrac{1}{2}\sigma_g^2\right) s_{g,t} \mathrm{d}t + \sigma_g s_{g,t} \mathrm{d}B_{g,t}$$

Note that using the household's budget constraint, we get in equilibrium:

$$\begin{aligned} \mathrm{d}a_t &= ((\alpha_t - \pi_t)a_t - c_t - g_t + y_t)\mathrm{d}t \\ &= (\alpha_t - \pi_t)a_t \mathrm{d}t, \end{aligned}$$

where for $\mathrm{d}a_t = 0$ either $\alpha_t = \pi_t$ and/or $a_t = 0$ for all t (we use $a_t = 0$ and $\alpha_t = 0$).

Moreover, in equilibrium the law of motion for the discounted expected future profits, $x_{1,t}$ and discounted expected future costs $x_{2,t}$ are *not* a function of the control variables:

$$\begin{aligned} \mathrm{d}x_{1,t} &= \left(((x_{2,t}/x_{1,t})\varepsilon/(\varepsilon-1))^{1-\varepsilon} \delta x_{1,t} - \lambda_t y_t\right) \mathrm{d}t \\ &= \left(((x_{2,t}/x_{1,t})\varepsilon/(\varepsilon-1))^{1-\varepsilon} \delta x_{1,t} - d_t/(1 - s_g s_{g,t})\right) \mathrm{d}t \end{aligned}$$

and similarly:

$$\begin{aligned} \mathrm{d}x_{2,t} &= \left(\left(\varepsilon\left((x_{2,t}/x_{1,t})\varepsilon/(\varepsilon-1)\right)^{1-\varepsilon} - 1\right) \delta x_{2,t}/(\varepsilon-1) - \lambda_t y_t mc_t\right) \mathrm{d}t \\ &= \left(((x_{2,t}/x_{1,t})\varepsilon/(\varepsilon-1))^{1-\varepsilon} \delta x_{2,t}\varepsilon/(\varepsilon-1) - \delta x_{2,t}/(\varepsilon-1) - mc_t d_t/(1 - s_g s_{g,t})\right) \mathrm{d}t \end{aligned}$$

Note that the TVC requires that $\lim_{t\to\infty} e^{-\rho t}\mathbb{E}_0 V(\mathbb{Z}_t^*) = 0$, in which $\mathbb{Z}_t^*$ denotes the state variables along the optimal path in line with general equilibrium conditions.

A.4. Proof of Proposition 1

We insert $\mathrm{d}c_t$ from (32) and the law of motions for the state variables

$$
\begin{gathered}
-(\rho - i_t + \pi_t)c_t\mathrm{d}t + \sigma_d^2\frac{d_t^2}{c_t}c_d^2\mathrm{d}t + \sigma_A^2\frac{A_t^2}{c_t}c_A^2\mathrm{d}t + \sigma_g^2\frac{s_{g,t}^2}{c_t}c_g^2\mathrm{d}t + \sigma_m^2\frac{1}{c_t}c_r^2\mathrm{d}t \\
+\sigma_d c_d d_t \mathrm{d}B_{d,t} + \sigma_A A_t c_A \mathrm{d}B_{A,t} + \sigma_g s_{g,t} c_g \mathrm{d}B_{g,t} + \sigma_m c_r \mathrm{d}B_{m,t} \\
-c_t\rho_d \log d_t \mathrm{d}t + \tfrac{1}{2}c_t\sigma_d^2\mathrm{d}t - \sigma_d^2 d_t c_d \mathrm{d}t \\
-\tfrac{1}{2}c_{rr}\sigma_m^2\,\mathrm{d}t - \tfrac{1}{2}c_{dd}(\sigma_d d_t)^2\mathrm{d}t - \tfrac{1}{2}c_{AA}(\sigma_A A_t)^2\mathrm{d}t - \tfrac{1}{2}c_{gg}(\sigma_g s_{g,t})^2\mathrm{d}t = \\
c_a\left((i_t - \pi_t)a_t - c_t + w_t l_t + T_t + F_t\right)\mathrm{d}t \\
+c_r((\theta\phi_\pi(\pi_t - \pi_{ss}) + \theta\phi_y(y_t/y_{ss} - 1) - \theta(i_t - i_{ss}))\mathrm{d}t + \sigma_m \mathrm{d}B_{m,t}) \\
+c_v(\delta(1 + \pi_t(1-\varepsilon)/\delta)^{\frac{\varepsilon}{1-\varepsilon}} + (\varepsilon\pi_t - \delta)v_t)\mathrm{d}t \\
+c_A(-\left(\rho_A \log A_t - \tfrac{1}{2}\sigma_A^2\right)A_t\mathrm{d}t + \sigma_A A_t \mathrm{d}B_{A,t}) \\
+c_d(-\left(\rho_d \log d_t - \tfrac{1}{2}\sigma_d^2\right)d_t\mathrm{d}t + \sigma_d d_t \mathrm{d}B_{d,t}) \\
+c_g(-\left(\rho_g \log s_{g,t} - \tfrac{1}{2}\sigma_g^2\right)s_{g,t}\mathrm{d}t + \sigma_g s_{g,t}\mathrm{d}B_{g,t})
\end{gathered}
$$

Collecting terms together with

$$c_a = -d_t V_a^{-2} V_{aa}$$

$$c_r = -d_t V_a^{-2} V_{ar}, \quad c_{rr} = 2d_t V_a^{-3} V_{ar}^2 - d_t V_a^{-2} V_{arr},$$

$$c_v = -d_t V_a^{-2} V_{av}$$

$$c_d = V_a^{-1} - d_t V_a^{-2} V_{ad}, \quad c_{dd} = -2V_a^{-2} V_{ad} + 2d_t V_a^{-3} V_{ad}^2 - d_t V_a^{-2} V_{add},$$

$$c_A = -d_t V_a^{-2} V_{aA}, \quad c_{AA} = 2d_t V_a^{-3} V_{aA}^2 - d_t V_a^{-2} V_{aAA}$$

$$c_g = -d_t V_a^{-2} V_{ag}, \quad c_{gg} = 2d_t V_a^{-3} V_{ag}^2 - d_t V_a^{-2} V_{agg}$$

we may eliminate time (and stochastic shocks) and arrive at

$$
\begin{gathered}
-(\rho - i_t + \pi_t) d_t V_a^{-1} \mathrm{d}t \\
+\sigma_d^2 \frac{d_t^2}{d_t V_a^{-1}} \left(V_a^{-2} - 2 d_t V_a^{-2} V_a^{-1} V_{ad} + d_t^2 V_a^{-4} V_{ad}^2 \right) \mathrm{d}t \\
+\sigma_A^2 \frac{A_t^2}{d_t V_a^{-1}} d_t^2 V_a^{-4} V_{aA}^2 \mathrm{d}t + \sigma_g^2 \frac{s_{g,t}^2}{d_t V_a^{-1}} d_t^2 V_a^{-4} V_{ag}^2 \mathrm{d}t + \sigma_m^2 \frac{1}{d_t V_a^{-1}} d_t^2 V_a^{-4} V_{ar}^2 \mathrm{d}t \\
+\sigma_d (V_a^{-1} - d_t V_a^{-2} V_{ad}) d_t \mathrm{d}B_{d,t} - \sigma_A A_t d_t V_a^{-2} V_{aA} \mathrm{d}B_{A,t} - \sigma_g s_{g,t} d_t V_a^{-2} V_{ag} \mathrm{d}B_{g,t} \\
-\sigma_m d_t V_a^{-2} V_{ar} \mathrm{d}B_{m,t} - d_t V_a^{-1} \rho_d \log d_t \mathrm{d}t + \tfrac{1}{2} d_t V_a^{-1} \sigma_d^2 \mathrm{d}t \\
-\sigma_d^2 d_t (V_a^{-1} - d_t V_a^{-2} V_{ad}) \mathrm{d}t - \tfrac{1}{2} \left(2 d_t V_a^{-3} V_{ar}^2 - d_t V_a^{-2} V_{arr} \right) \sigma_m^2 \, \mathrm{d}t \\
-\tfrac{1}{2} \left(-2 V_a^{-2} V_{ad} + 2 d_t V_a^{-3} V_{ad}^2 - d_t V_a^{-2} V_{add} \right) (\sigma_d d_t)^2 \mathrm{d}t \\
-\tfrac{1}{2} \left(2 d_t V_a^{-3} V_{aA}^2 - d_t V_a^{-2} V_{aAA} \right) (\sigma_A A_t)^2 \mathrm{d}t \\
-\tfrac{1}{2} \left(2 d_t V_a^{-3} V_{ag}^2 - d_t V_a^{-2} V_{agg} \right) (\sigma_g s_{g,t})^2 \mathrm{d}t = \\
-d_t V_a^{-2} V_{aa} \left((i_t - \pi_t) a_t - c_t + w_t l_t + T_t + \digamma_t \right) \mathrm{d}t \\
-d_t V_a^{-2} V_{ar} ((\theta \phi_\pi (\pi_t - \pi_{ss}) + \theta \phi_y (y_t / y_{ss} - 1) - \theta (i_t - i_{ss})) \mathrm{d}t + \sigma_m \mathrm{d}B_{m,t}) \\
-d_t V_a^{-2} V_{av} (\delta (1 + \pi_t (1 - \varepsilon) / \delta)^{\frac{\varepsilon}{1-\varepsilon}} + (\varepsilon \pi_t - \delta) v_t) \mathrm{d}t \\
-d_t V_a^{-2} V_{aA} (- \left(\rho_A \log A_t - \tfrac{1}{2} \sigma_A^2 \right) A_t \mathrm{d}t + \sigma_A A_t \mathrm{d}B_{A,t}) \\
+(V_a^{-1} - d_t V_a^{-2} V_{ad}) (- \left(\rho_d \log d_t - \tfrac{1}{2} \sigma_d^2 \right) d_t \mathrm{d}t + \sigma_d d_t \mathrm{d}B_{d,t}) \\
-d_t V_a^{-2} V_{ag} (- \left(\rho_g \log s_{g,t} - \tfrac{1}{2} \sigma_g^2 \right) s_{g,t} \mathrm{d}t + \sigma_g s_{g,t} \mathrm{d}B_{g,t})
\end{gathered}
$$

which can be simplified to

$$
\begin{gathered}
-(\rho - i_t + \pi_t) V_a \mathrm{d}t = \\
- \left((i_t - \pi_t) a_t - c_t + w_t l_t + T_t + \digamma_t \right) V_{aa} \mathrm{d}t \\
-(\theta \phi_\pi (\pi_t - \pi_{ss}) + \theta \phi_y (y_t / y_{ss} - 1) - \theta (i_t - i_{ss})) V_{ar} \mathrm{d}t - \tfrac{1}{2} V_{arr} \sigma_m^2 \, \mathrm{d}t \\
-(\delta (1 + \pi_t (1 - \varepsilon) / \delta)^{\frac{\varepsilon}{1-\varepsilon}} + (\varepsilon \pi_t - \delta) v_t) V_{av} \mathrm{d}t \\
+ \left(\rho_A \log A_t - \tfrac{1}{2} \sigma_A^2 \right) A_t V_{aA} \mathrm{d}t - \tfrac{1}{2} V_{aAA} (\sigma_A A_t)^2 \mathrm{d}t \\
+V_{ad} \left(\rho_d \log d_t - \tfrac{1}{2} \sigma_d^2 \right) d_t \mathrm{d}t - \tfrac{1}{2} V_{add} \sigma_d^2 d_t^2 \mathrm{d}t \\
+V_{ag} \left(\rho_g \log s_{g,t} - \tfrac{1}{2} \sigma_g^2 \right) s_{g,t} \mathrm{d}t - \tfrac{1}{2} V_{agg} (\sigma_g s_{g,t})^2 \mathrm{d}t
\end{gathered}
$$

such that (36) must hold as an identity.

A.5. Analytical results

Steady-state. Suppose that without shocks the economy moves towards its steady state. Setting the variance of shocks to zero yields the deterministic steady state values.

- Euler equation, the first-order conditions of the household, and budget constraint:

  Equation 1

$$\pi_{ss} = i_{ss} - \rho$$

  Equation 2

$$\psi l_{ss}^{\vartheta} c_{ss} = w_{ss}$$

  Equation 3

$$d_{ss} c_{ss}^{-1} = \lambda_{ss}$$

- Profit maximization is given by:

  Equation 4

$$\Pi^*_{ss} = \frac{\varepsilon}{\varepsilon - 1} \frac{x_{2,ss}}{x_{1,ss}}$$

  Equation 5

$$0 = (\rho + \delta - (\varepsilon - 1)\pi_{ss})x_{1,ss} - \lambda_{ss} y_{ss}$$

  Equation 6

$$0 = (\rho + \delta - \varepsilon \pi_t)\, x_{2,ss} - \lambda_{ss} y_{ss} mc_{ss}$$

  Equation 7

$$F_{ss} = (1 - mc_{ss} v_{ss}) y_{ss}$$

  Equation 8

$$w_{ss} = A_{ss} mc_{ss}$$

- Government policy:

  Equation 9

  (This equation is an identity in the steady state.)

  Equation 10

$$g_{ss} = s_g s_{g,ss} y_{ss}$$

- Inflation evolution and price dispersion:

  Equation 11

$$\pi_{ss} = \frac{\delta}{1 - \varepsilon} \left((\Pi^*_{ss})^{1-\varepsilon} - 1 \right)$$

  Equation 12

$$0 = \delta \left(\Pi^*_{ss}\right)^{-\varepsilon} + (\varepsilon \pi_{ss} - \delta) v_{ss}$$

- Market clearing on goods and labor markets (one condition is redundant):

Equation 13

$$y_{ss} = c_{ss} + g_{ss} \quad \text{(expenditure)}$$

Equation 14

$$y_{ss} = \frac{A_{ss}}{v_{ss}} l_{ss} \quad \text{(production)}$$

(redundant)

$$y_{ss} = w_{ss} l_{ss} + F_{ss} \quad \text{(income)}$$

- Stochastic processes:

Equation 15

$$d_{ss} = 1$$

Equation 16

$$A_{ss} = 1$$

Equation 17

$$s_{g,ss} = 1$$

Given the level of steady-state inflation, around which the model often is (log-)linearized, we obtain the following steady-state values. Using **Equation 1**, we obtain:

$$i_{ss} = \pi_{ss} + \rho$$

Using **Equation 11**, we obtain the steady-state value for the price ratio:

$$\Pi^*_{ss} = (1 - (\varepsilon - 1)(\pi_{ss}/\delta))^{\frac{1}{1-\varepsilon}}$$

Using **Equations 5** and **6** we can solve for the steady-state value of the marginal cost:

$$mc_{ss} = (\rho + \delta - \varepsilon \pi_{ss})\,(1 - s_g s_{g,ss}) x_{2,ss}/d_{ss}$$

or

$$mc_{ss} = \frac{(\rho + \delta - \varepsilon \pi_{ss})}{(\rho + \delta - (\varepsilon - 1)\pi_{ss})} (x_{2,ss}/x_{1,ss})$$

which by inserting **Equation 4** gives:

$$mc_{ss} = \frac{\rho + \delta - \varepsilon \pi_{ss}}{\rho + \delta - (\varepsilon - 1)\pi_{ss}} \frac{\varepsilon - 1}{\varepsilon} \Pi^*_{ss}$$

Hence, we obtain

$$x_{1,ss} = d_{ss}/((1 - s_g s_{g,ss})(\rho + \delta - (\varepsilon - 1)\pi_{ss}))$$

and

$$x_{2,ss} = (1 - 1/\varepsilon)x_{1,ss}\Pi^*_{ss}$$

Using **Equation 8**, we obtain

$$w_{ss} = A_{ss}mc_{ss}$$

Using **Equation 12** gives the steady-state value of price dispersion

$$v_{ss} = \frac{\delta\left(\Pi^*_{ss}\right)^{-\varepsilon}}{\delta - \varepsilon\pi_{ss}}$$

Using **Equation 14**, we obtain

$$y_{ss} = A_{ss}l_{ss}/v_{ss}$$

Using **Equation 13** and **Equation 10** yields

$$y_{ss} = c_{ss}/(1 - s_g s_{g,ss})$$

Combining the last two equations gives

$$A_{ss}l_{ss}/v_{ss} = c_{ss}/(1 - s_g s_{g,ss})$$

Using **Equation 2** we get

$$\psi l^{\vartheta}_{ss}c_{ss} = w_{ss}$$

hence we can collect terms to obtain

$$l_{ss} = \left(\frac{w_{ss}v_{ss}}{\psi(1 - s_g s_{g,ss})A_{ss}}\right)^{\frac{1}{1+\vartheta}}$$

Using **Equation 7** and **Equation 14** we get

$$F_{ss} = (1 - mc_{ss}v_{ss})A_{ss}l_{ss}/v_{ss}$$

The deterministic values, however, do not necessarily correspond to the stationary points in the absence of shocks, i.e., the values at which the variables are expected to stay idle in the presence of risk. Hence, the stochastic steady state values are obtained from the conditional deterministic equations, setting the random shocks (not their variances) to zero. We may thus start with (7) and compute $\mathbb{E}(\,\mathrm{d}d_t) = 0$, or

$$0 \;=\; -\left(\rho_d \log d_t - \tfrac{1}{2}\sigma_d^2\right) d_t \mathrm{d}t \quad\Rightarrow\quad d_{ss} = \exp(\tfrac{1}{2}\sigma_d^2/\rho_d)$$

The stochastic steady state values are similar to – but do not necessarily reflect – the first moments of the variables. For example, the preference shock implies:

$$\mathrm{d}\log d_t = -\rho_d \log d_t \mathrm{d}t + \sigma_d \mathrm{d}B_{d,t} \;\Leftrightarrow\; \log d_t = e^{-\rho_d t}\log d_0 + \sigma_d \int_0^t e^{\rho(s-t)}\,\mathrm{d}B_s,$$

which has a long-run (or stationary) Normal distribution $\log d_t \sim \mathcal{N}(0, \frac{1}{2}\sigma_d^2/\rho_d)$.[30] Hence, if $\log d_t$ is asymptotically normally distributed, the variable d_t is log-normally distributed $d_t \sim \mathcal{LN}(0, \frac{1}{2}\sigma^2/\rho_d)$ with expected value

$$\mathbb{E}(d_t) = \exp(\tfrac{1}{4}\sigma_d^2/\rho_d)$$

Similarly, we obtain the stochastic steady states for

$$\begin{aligned} 0 &= -\left(\rho_A \log A_t - \tfrac{1}{2}\sigma_A^2\right) A_t \mathrm{d}t \quad\Rightarrow\quad A_{ss} = \exp(\tfrac{1}{2}\sigma_A^2/\rho_A) \\ 0 &= -\left(\rho_g \log s_{g,t} - \tfrac{1}{2}\sigma_g^2\right) s_{g,t} \mathrm{d}t \quad\Rightarrow\quad s_{g,ss} = \exp(\tfrac{1}{2}\sigma_g^2/\rho_g) \end{aligned}$$

It shows that the unconditional mean value of the stationary distribution roughly coincides with our notion of a stochastic steady state. But even if the process started at its long-run stationary mean value $d_0 = \frac{1}{4}\sigma_d^2/\rho_d$, our derivations show that the infinitesimal change in the variable d_t at $t = 0$ is expected to be positive, only at the value $d_0 = \frac{1}{2}\sigma_d^2/\rho_d$, the process is expected to stay at the same value such that $\mathbb{E}(\,\mathrm{d}d_t) = 0$ at $t = 0$.

Steady-state. In the presence of uncertainty, in case the dynamic variables approach a stochastic steady-state distribution (a stationary distribution), they also approach their stochastic steady-state values (in the absence of further shocks). Hence, for a given inflation target π_{ss}, the Euler equation (37) determines the long-run value i_{ss}.

[30] The moments of the stationary distribution can be obtained from

$$\begin{aligned} \mathrm{d}(\log d_t)^2 &= 2\log d_t \,\mathrm{d}\log d_t + \sigma_d^2\,\mathrm{d}t \\ &= -\rho_d 2\log d_t \log d_t \mathrm{d}t + \sigma_d 2\log d_t \mathrm{d}B_{d,t} + \sigma_d^2\,\mathrm{d}t \end{aligned}$$

the expected value reads

$$\mathrm{d}\mathbb{E}(\log d_t) = -\rho_d \mathrm{d}\mathbb{E}(\log d_t)\mathrm{d}t \;\Leftrightarrow\; \mathbb{E}(\log d_t) = e^{-\rho_d t}\log d_0 \quad\Rightarrow\quad \lim_{t\to\infty}\mathbb{E}(\log d_t) = 0$$

and

$$\mathrm{d}\mathbb{E}((\log d_t)^2) = -\rho_d 2\mathbb{E}((\log d_t)^2)\mathrm{d}t + \sigma_d^2\,\mathrm{d}t$$

such that

$$\mathbb{E}((\log d_t)^2) = \mathbb{V}ar((\log d_t)^2) = \tfrac{1}{2}\sigma_d^2/\rho_d$$

- Euler equation, and the first-order conditions of the household:

Equation 1

$$\pi_{ss} = i_{ss} - \rho + \tilde{c}_d^2\sigma_d^2 + \tilde{c}_A^2\sigma_A^2 + \tilde{c}_g^2\sigma_g^2 + \tilde{c}_r^2\sigma_m^2 - \frac{1}{2}\tilde{c}_{dd}\sigma_d^2 - \frac{1}{2}\tilde{c}_{AA}\sigma_A^2 - \frac{1}{2}\tilde{c}_{gg}\sigma_g^2 - \frac{1}{2}\tilde{c}_{rr}\sigma_m^2 - \tilde{c}_d\sigma_d^2$$

Equation 2

$$\psi l_{ss}^{\vartheta} c_{ss} = w_{ss}$$

Equation 3

$$d_{ss} c_{ss}^{-1} = \lambda_{ss}$$

- Profit maximization is given by:

Equation 4

$$\Pi_{ss}^{*} = \frac{\varepsilon}{\varepsilon - 1}\frac{x_{2,ss}}{x_{1,ss}}$$

Equation 5

$$0 = (\rho + \delta - (\varepsilon - 1)\pi_{ss})x_{1,ss} - \lambda_{ss} y_{ss}$$

Equation 6

$$0 = (\rho + \delta - \varepsilon\pi_t)\, x_{2,ss} - \lambda_{ss} y_{ss} mc_{ss}$$

Equation 7

$$\digamma_{ss} = (1 - mc_{ss} v_{ss}) y_{ss}$$

Equation 8

$$w_{ss} = A_{ss} mc_{ss}$$

- Government policy:

Equation 9

(This equation is an identity in the steady state.)

Equation 10

$$g_{ss} = s_g s_{g,ss} y_{ss}$$

- Inflation evolution and price dispersion:

Equation 11

$$\pi_{ss} = \frac{\delta}{1 - \varepsilon}\left(\left(\Pi_{ss}^{*}\right)^{1-\varepsilon} - 1\right)$$

Equation 12

$$0 = \delta\left(\Pi_{ss}^{*}\right)^{-\varepsilon} + (\varepsilon\pi_{ss} - \delta) v_{ss}$$

- Market clearing on goods and labor markets (one condition is redundant):

Equation 13

$$y_{ss} = c_{ss} + g_{ss} \quad \text{(expenditure)}$$

Equation 14

$$y_{ss} = \frac{A_{ss}}{v_{ss}} l_{ss} \quad \text{(production)}$$

(redundant)

$$y_{ss} = w_{ss} l_{ss} + F_{ss} \quad \text{(income)}$$

- Stochastic processes:

Equation 15

$$d_{ss} = \exp(\tfrac{1}{2}\sigma_d^2/\rho_d)$$

Equation 16

$$A_{ss} = \exp(\tfrac{1}{2}\sigma_A^2/r_A)$$

Equation 17

$$s_{g,ss} = \exp(\tfrac{1}{2}\sigma_g^2/r_g)$$

Given an inflation target π_{ss}, **Equation 1** pins down the long-run interest rate as

$$i_{ss} = \pi_{ss} + \rho - (\tilde{c}_d^2\sigma_d^2 + \tilde{c}_A^2\sigma_A^2 + \tilde{c}_g^2\sigma_g^2 + \tilde{c}_r^2\sigma_m^2 - \tfrac{1}{2}\tilde{c}_{dd}\sigma_d^2 - \tfrac{1}{2}\tilde{c}_{AA}\sigma_A^2 - \tfrac{1}{2}\tilde{c}_{gg}\sigma_g^2 - \tfrac{1}{2}\tilde{c}_{rr}\sigma_m^2 - \tilde{c}_d\sigma_d^2)$$

or

$$i_{ss} = \pi_{ss} + \rho + RP_{ss}$$

Using **Equation 11**, we obtain the steady-state value Π^*_{ss}

$$\Pi^*_{ss} = (1 - (\varepsilon - 1)(\pi_{ss}/\delta))^{\frac{1}{1-\varepsilon}}$$

Using **Equations 5** and **6** we can solve for the steady-state value of the marginal cost:

$$mc_{ss} = (\rho + \delta - \varepsilon\pi_{ss})\,(1 - s_g s_{g,ss}) x_{2,ss}/d_{ss}$$

or

$$mc_{ss} = \frac{\rho + \delta - \varepsilon\pi_{ss}}{\rho + \delta - (\varepsilon - 1)\pi_{ss}} \frac{x_{2,ss}}{x_{1,ss}}$$

which by inserting **Equation 4** gives:

$$mc_{ss} = \frac{\rho + \delta - \varepsilon \pi_{ss}}{\rho + \delta - (\varepsilon - 1)\pi_{ss}} \frac{\varepsilon - 1}{\varepsilon} \Pi^*_{ss}$$

Hence, we obtain

$$x_{1,ss} = d_{ss}/((1 - s_g s_{g,ss})(\rho + \delta - (\varepsilon - 1)\pi_{ss}))$$

and

$$x_{2,ss} = (1 - 1/\varepsilon) x_{1,ss} \Pi^*_{ss}$$

Using **Equation 8**, we obtain

$$w_{ss} = A_{ss} mc_{ss}$$

Using **Equation 12** gives the steady-state value of price dispersion

$$v_{ss} = \frac{\delta \left(\Pi^*_{ss}\right)^{-\varepsilon}}{\delta - \varepsilon \pi_{ss}}$$

Using **Equation 14**, we obtain

$$y_{ss} = A_{ss} l_{ss} / v_{ss}$$

Using **Equation 13** and **Equation 10** yields

$$y_{ss} = c_{ss}/(1 - s_g s_{g,ss})$$

Combining the last two equations gives

$$A_{ss} l_{ss} / v_{ss} = c_{ss}/(1 - s_g s_{g,ss})$$

Using **Equation 2** we get

$$\psi l^{\vartheta}_{ss} c_{ss} = w_{ss}$$

hence we can collect terms to obtain

$$l_{ss} = \left(\frac{w_{ss} v_{ss}}{\psi (1 - s_g s_{g,ss}) A_{ss}} \right)^{\frac{1}{1+\vartheta}}$$

Using **Equation 7** and **Equation 14** we get

$$F_{ss} = (1 - mc_{ss} v_{ss}) A_{ss} l_{ss} / v_{ss}$$

A.6. Linear Approximations

In order to analyze local dynamics, the traditional approach is to approximate the dynamic equilibrium system around steady-state values. We define we $\hat{x}_t \equiv (x_t - x_{ss})/x_{ss}$, where x_{ss} is the steady-state value for the variable x_t. Thus, we can write $x_t = (1 + \hat{x}_t)x_{ss}$.[31]

- Euler equation, the first-order conditions of the household, and budget constraint:

Equation 1

$$\mathrm{d}(c_t/c_{ss} - 1) = -(\rho - i_t + \pi_t + \rho_d(d_t/d_{ss} - 1))\mathrm{d}t$$

Equation 2

$$c_t/c_{ss} + \vartheta(l_t/l_{ss} - 1) = w_t/w_{ss}$$

Equation 3

$$d_t/d_{ss} - c_t/c_{ss} = \lambda_t/\lambda_{ss} - 1$$

- Profit maximization is given by:

Equation 4

$$\hat{\Pi}_t^* = \hat{x}_{2,t} - \hat{x}_{1,t}$$

Equation 5

$$\begin{aligned}\mathrm{d}(x_{1,t}/x_{1,ss} - 1) &= ((\rho + a_2)(x_{1,t}/x_{1,ss} - 1) - (\varepsilon - 1)(\pi_t - \pi_{ss}))\,\mathrm{d}t \\ &-y_{ss}(d_{ss}/c_{ss})\left((y_t/y_{ss} - 1) + (d_t/d_{ss} - 1) - (c_t/c_{ss} - 1)\right)/x_{1,ss}\,\mathrm{d}t\end{aligned}$$

Equation 6

$$\mathrm{d}(x_{2,t}/x_{2,ss} - 1) = (a_1(x_{2,t}/x_{2,ss} - 1) - \varepsilon(\pi_t - \pi_{ss}))\,\mathrm{d}t$$
$$-mc_{ss}y_{ss}(d_{ss}/c_{ss})\left((mc_t/mc_{ss} - 1) + (y_t/y_{ss} - 1) + (d_t/d_{ss} - 1) - (c_t/c_{ss} - 1)\right)/x_{2,ss}\,\mathrm{d}t$$

Equation 7

$$(1 - mc_{ss}v_{ss})(\digamma_t/\digamma_{ss} - 1) = (1 - mc_{ss}v_{ss})(y_t/y_{ss} - 1) - mc_{ss}v_{ss}(mc_t/mc_{ss} + v_t/v_{ss})$$

Equation 8

$$w_t/w_{ss} - 1 = A_t/A_{ss} + mc_t/mc_{ss}$$

[31] In what follows we (log-)linearize around non-stochastic steady-state values, in particular, we assume certainty equivalence (as an approximation), which amounts to setting $\sigma_d^2 = \sigma_A^2 = \sigma_d^2 = \sigma_m^2 = 0$.

- Government policy:

Equation 9

$$\mathrm{d}(i_t - i_{ss}) = (\theta\phi_\pi(\pi_t - \pi_{ss}) + \theta\phi_y(y_t/y_{ss} - 1) - \theta(i_t - i_{ss}))\,\mathrm{d}t$$

Equation 10

$$g_t/g_{ss} - 1 = s_{g,t}/s_{g,ss} + y_t/y_{ss}$$

- Inflation evolution and price dispersion:

Equation 11

$$\pi_t - \pi_{ss} = (\delta + (1-\varepsilon)\pi_{ss})(x_{2,t}/x_{2,ss} - x_{1,t}/x_{1,ss})$$

Equation 12

$$\mathrm{d}(v_t/v_{ss} - 1) = \frac{\varepsilon\pi_{ss}}{\delta + (1-\varepsilon)\pi_{ss}}(\pi_t - \pi_{ss})\mathrm{d}t + (\varepsilon\pi_{ss} - \delta)(v_t/v_{ss} - 1)\mathrm{d}t$$

- Market clearing on goods and labor markets:

Equation 13

$$y_t/y_{ss} = c_t/c_{ss} + s_g s_{g,ss}/(1 - s_g s_{g,ss})(s_{g,t}/s_{g,ss} - 1)$$

Equation 14

$$y_t/y_{ss} = A_t/A_{ss} + l_t/l_{ss} - v_t/v_{ss}$$

- Stochastic processes follow:

Equation 15

$$\mathrm{d}(d_t/d_{ss} - 1) = -\rho_d(d_t/d_{ss} - 1)\mathrm{d}t$$

Equation 16

$$\mathrm{d}(A_t/A_{ss} - 1) = -\rho_A(A_t/A_{ss} - 1)\mathrm{d}t$$

Equation 17

$$\mathrm{d}(s_{g,t}/s_{g,ss} - 1) = -\rho_g(s_{g,t}/s_{g,ss} - 1)\mathrm{d}t$$

Hence, we may summarize the local equilibrium dynamics around steady-state values as:

$$
\begin{aligned}
\mathrm{d}i_t &= \theta(\phi_\pi a_2(\hat{x}_{2,t} - \hat{x}_{1,t}) + \phi_y(\hat{c}_t + s_g s_{g,ss}/(1 - s_g s_{g,ss})\hat{s}_{g,t}) - (i_t - i_{ss}))\,\mathrm{d}t \\
\mathrm{d}\hat{v}_t &= \varepsilon\pi_{ss}(\hat{x}_{2,t} - \hat{x}_{1,t})\mathrm{d}t + (\varepsilon\pi_{ss} - \delta)\hat{v}_t\mathrm{d}t \\
\mathrm{d}\hat{d}_t &= -\rho_d\hat{d}_t\mathrm{d}t \\
\mathrm{d}\hat{A}_t &= -\rho_A\hat{A}_t\mathrm{d}t \\
\mathrm{d}\hat{s}_{g,t} &= -\rho_g\hat{s}_{g,t}\mathrm{d}t \\
\mathrm{d}\hat{x}_{1,t} &= ((\rho + \varepsilon a_2)\hat{x}_{1,t} - (\varepsilon - 1)a_2\hat{x}_{2,t} - y_{ss}(d_{ss}/c_{ss})(s_g s_{g,ss}/(1 - s_g s_{g,ss})\hat{s}_{g,t} + \hat{d}_t)/x_{1,ss})\mathrm{d}t \\
\mathrm{d}\hat{x}_{2,t} &= (a_1\hat{x}_{2,t} - \varepsilon a_2(\hat{x}_{2,t} - \hat{x}_{1,t}))\,\mathrm{d}t \\
&\quad -a_1((1 + \vartheta)(s_g s_{g,ss}/(1 - s_g s_{g,ss})\hat{s}_{g,t} + \hat{c}_t - \hat{A}_t) + \vartheta\hat{v}_t + \hat{d}_t)\mathrm{d}t \\
\mathrm{d}\hat{c}_t &= ((i_t - i_{ss}) - a_2(\hat{x}_{2,t} - \hat{x}_{1,t}) - \rho_d\hat{d}_t)\mathrm{d}t
\end{aligned}
$$

in which we define percentage deviations $\hat{x}_t \equiv (x_t - x_{ss})/x_{ss}$ and used the definitions for $a_1 \equiv \rho + \delta - \varepsilon\pi_{ss}$, and $a_2 \equiv \delta + (1 - \varepsilon)\pi_{ss}$ in the main text.

In order to analyze local dynamics around the non-stochastic steady state, we need to study the eigenvalues of the Jacobian matrix evaluated at the steady state:

$$
\mathrm{d}\begin{bmatrix} i_t - i_{ss} \\ \hat{v}_t \\ \hat{d}_t \\ \hat{A}_t \\ \hat{s}_{g,t} \\ \hat{x}_{1,t} \\ \hat{x}_{2,t} \\ \hat{c}_t \end{bmatrix} = \begin{bmatrix} a_{11} & 0 & 0 & 0 & a_{15} & a_{16} & a_{17} & a_{18} \\ 0 & a_{22} & 0 & 0 & 0 & a_{26} & a_{27} & 0 \\ 0 & 0 & a_{33} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{44} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{55} & 0 & 0 & 0 \\ 0 & 0 & a_{63} & 0 & a_{65} & a_{66} & a_{67} & 0 \\ 0 & a_{72} & a_{73} & a_{74} & a_{75} & a_{76} & a_{77} & a_{78} \\ a_{81} & 0 & a_{83} & 0 & 0 & a_{86} & a_{87} & 0 \end{bmatrix} \begin{bmatrix} i_t - i_{ss} \\ \hat{v}_t \\ \hat{d}_t \\ \hat{A}_t \\ \hat{s}_{g,t} \\ \hat{x}_{1,t} \\ \hat{x}_{2,t} \\ \hat{c}_t \end{bmatrix} \mathrm{d}t
$$

$$
\begin{aligned}
\text{where} \quad a_{11} &\equiv -\theta \\
a_{15} &\equiv \theta\phi_y s_g s_{g,ss}/(1 - s_g s_{g,ss}) \\
a_{16} &\equiv -\theta\phi_\pi a_2 \\
a_{17} &\equiv \theta\phi_\pi a_2 \\
a_{18} &\equiv \theta\phi_y \\
a_{22} &\equiv \varepsilon\pi_{ss} - \delta \\
a_{26} &\equiv -\varepsilon\pi_{ss} \\
a_{27} &\equiv \varepsilon\pi_{ss} \\
a_{33} &\equiv -\rho_d \\
a_{44} &\equiv -r_A \\
a_{55} &\equiv -r_g \\
a_{63} &\equiv -y_{ss}(d_{ss}/c_{ss})/x_{1,ss} \\
a_{65} &\equiv -y_{ss}(d_{ss}/c_{ss})s_g s_{g,ss}/(1 - s_g s_{g,ss})/x_{1,ss} \\
a_{66} &\equiv \rho + \varepsilon a_2 \\
a_{67} &\equiv -(\varepsilon - 1)a_2 \\
a_{72} &\equiv -a_1\vartheta \\
a_{73} &\equiv -a_1 \\
a_{74} &\equiv a_1(1+\vartheta) \\
a_{75} &\equiv -a_1(1+\vartheta)s_g s_{g,ss}/(1 - s_g s_{g,ss}) \\
a_{76} &\equiv \varepsilon a_2 \\
a_{77} &\equiv a_1 - \varepsilon a_2 \\
a_{78} &\equiv -a_1(1+\vartheta) \\
a_{81} &\equiv 1 \\
a_{83} &\equiv -\rho_d \\
a_{86} &\equiv a_2 \\
a_{87} &\equiv -a_2
\end{aligned}
$$

Recall that from (35) we may obtain the linearized static condition (not necessary):

$$\hat{m}c_t = -(1+\vartheta)(\hat{A}_t - \hat{c}) + \vartheta(\hat{v}_t + s_g/(1-s_g)\hat{s}_{g,t})$$

and the output gap is defined as:

$$y_t/y^n - 1 \equiv y_{ss}/y^n(A_t/A_{ss} + l_t/l_{ss} - v_t/v_{ss}) - 1$$

where y_n is the level of potential output that would prevail under flexible prices.

A.7. New-Keynesian analysis

This section sheds some light on the implications of the NK model for both the IS-curve and the NK Phillips-curve. We compare the implications of the nonlinear model with the (log-)linear approximation of the model currently used in the literature.

We start with the NK forward-looking Phillips-curve, which from (19), the first-order condition $\lambda_t = d_t/c_t$, and the market-clearing condition $c_t = (1 - s_g s_{g,t})y_t$ reads:

$$\mathrm{d}\pi_t \quad = \quad -(\delta + (1-\varepsilon)\pi_t)\left(\pi_t + (mc_t/x_{2,t} - 1/x_{1,t})\, d_t/(1 - s_g s_{g,t})\right)\mathrm{d}t,$$

which together with $mc_t = \psi l_t^{1+\vartheta}(1 - s_g s_{g,t})/v_t$ and $l_t = y_t v_t/A_t$, among other variables, shows the response of inflation to the output gap.[32] Hence, the linearized Phillips-curve around (stochastic) steady-state values reads (see appendix Section A.5 for definitions)

$$\begin{aligned}
\mathrm{d}\pi_t \quad = \quad & -(\delta + (1-\varepsilon)\pi_{ss})(\pi_t - \pi_{ss})\mathrm{d}t \\
& -(\delta + (1-\varepsilon)\pi_{ss})\left(\rho + \delta - \varepsilon\pi_{ss}\right)\left(mc_t/mc_{ss} - 1\right)\mathrm{d}t \\
& -(\delta + (1-\varepsilon)\pi_{ss})\left(\rho + \delta + (1-\varepsilon)\pi_{ss}\right)\left(x_{1,t}/x_{1,ss} - 1\right)\mathrm{d}t \\
& +(\delta + (1-\varepsilon)\pi_{ss})\left(\rho + \delta - \varepsilon\pi_{ss}\right)\left(x_{2,t}/x_{2,ss} - 1\right)\mathrm{d}t \\
& +(\delta + (1-\varepsilon)\pi_{ss})\pi_{ss}(d_t/d_{ss} - 1)\mathrm{d}t \\
& +(\delta + (1-\varepsilon)\pi_{ss})\pi_{ss}\left(s_g s_{g,ss}/(1 - s_g s_{g,ss})\right)\left(s_{g,t}/s_{g,ss} - 1\right)\mathrm{d}t.
\end{aligned}$$

in which

$$\pi_t - \pi_{ss} = (\delta + (1-\varepsilon)\pi_{ss})(x_{2,t}/x_{2,ss} - x_{1,t}/x_{1,ss})$$

It shows in the NK Phillips-curve how the change in inflation depends on marginal costs. We may insert the linearized equation for marginal cost,

$$\begin{aligned}
mc_t/mc_{ss} - 1 \quad = \quad & (1+\vartheta)(y_t/y_{ss} - 1) - (1+\vartheta)(A_t/A_{ss} - 1) \\
& +\vartheta(v_t/v_{ss} - 1) - s_g s_{g,ss}/(1 - s_g s_{g,ss})(s_{g,t}/s_{g,ss} - 1) \\
= \quad & (1+\vartheta)(c_t/c_{ss} - 1) - (1+\vartheta)(A_t/A_{ss} - 1) \\
& +\vartheta(v_t/v_{ss} - 1) + \vartheta(s_g s_{g,ss}/(1 - s_g s_{g,ss}))(s_{g,t}/s_{g,ss} - 1)
\end{aligned}$$

where

$$y_t/y_{ss} = c_t/c_{ss} + (s_g s_{g,ss}/(1 - s_g s_{g,ss}))(s_{g,t}/s_{g,ss} - 1)$$

to obtain the NK Phillips-curve with respect to the output gap and/or consumption.

[32] In order to analyze local dynamics, the traditional approach is to (log-)linearize the variables. We define $\hat{x}_t \equiv (x_t - x_{ss})/x_{ss}$, where x_{ss} is the steady-state value for the variable x_t, such that $x_t = (1+\hat{x}_t)x_{ss}$

As shown, the marginal costs are related to consumption (or output gap), the prevailing level of price dispersion, the technology shock and the government expenditure shock. A description of the (local) dynamics includes the equation of price dispersion[33]

$$\mathrm{d}(v_t/v_{ss}-1) \;=\; \frac{\varepsilon\pi_{ss}}{\delta+(1-\varepsilon)\pi_{ss}}(\pi_t-\pi_{ss})\mathrm{d}t+(\varepsilon\pi_{ss}-\delta)(v_t/v_{ss}-1)\mathrm{d}t$$

From (40), the linearized Euler equation (of the conditional deterministic system) reads

$$\begin{aligned}\mathrm{d}(c_t/c_{ss}-1) &= (i_t-i_{ss}-(\pi_t-\pi_{ss})-(RP_t-RP_{ss})-\rho_d(d_t/d_{ss}-1))\mathrm{d}t\\ &= (i_t-\rho-\pi_t-RP_t-\rho_d(d_t/d_{ss}-1))\mathrm{d}t\end{aligned}$$

which is readily interpreted as the (micro-founded) NK IS-curve.

To summarize, the equilibrium dynamics of the linearized model around $\pi_{ss}=0$ are

$$\begin{aligned}\mathrm{d}(c_t/c_{ss}-1) &= (i_t-\rho-\pi_t-RP_t-\rho_d(d_t/d_{ss}-1))\mathrm{d}t\\ \mathrm{d}\pi_t &= (\rho(\pi_t-\pi_{ss})-\delta\,(\rho+\delta)\,(mc_t/mc_{ss}-1))\,\mathrm{d}t\\ \mathrm{d}i_t &= (\theta\phi_\pi(\pi_t-\pi_{ss})+\theta\phi_y(y_t/y_{ss}-1)-\theta(i_t-i_{ss}))\mathrm{d}t\\ \mathrm{d}(v_t/v_{ss}-1) &= -\delta(v_t/v_{ss}-1)\mathrm{d}t\end{aligned}$$

where the first equation denotes the IS-curve (or linearized Euler equation), the second the NK Phillips-curve (showing the dependence on marginal costs), and the third the Taylor rule reflecting the interest rate responds of the monetary authority.

The (linearized) system as in Werning (2012) and Cochrane (2017b) can be obtained by taking a long-term perspective (after transitional dynamics), by defining the natural rate $r_t\equiv\rho+RP_t+\rho_d(d_t/d_{ss}-1)$, assuming it to be exogenous, and $x_t\equiv(y_t/y_{ss}-1)$

$$\begin{aligned}\mathrm{d}x_t &= (i_t-r_t-\pi_t)\mathrm{d}t\\ \mathrm{d}\pi_t &= (\rho(\pi_t-\pi_{ss})-\delta\,(\rho+\delta)\,(1+\vartheta)x_t)\,\mathrm{d}t\\ \mathrm{d}i_t &= (\theta\phi_\pi(\pi_t-\pi_{ss})+\theta\phi_y x_t-\theta(i_t-i_{ss}))\mathrm{d}t\end{aligned}$$

where our specification of the Taylor rule is similar to Cochrane (2017b). We may restate

[33] The dynamics of the three shocks from the (conditional) deterministic system are

$$\begin{aligned}\mathrm{d}(d_t/d_{ss}-1) &= \left[-\left(\rho_d\log d_{ss}-\tfrac{1}{2}\sigma_d^2\right)-\rho_d\right](d_t/d_{ss}-1)\mathrm{d}t\\ &= -\rho_d(d_t/d_{ss}-1)\mathrm{d}t\\ \mathrm{d}(A_t/A_{ss}-1) &= -\rho_A(A_t/A_{ss}-1)\mathrm{d}t\\ \mathrm{d}(s_{g,t}/s_{g,ss}-1) &= -\rho_g(s_{g,t}/s_{g,ss}-1)\mathrm{d}t\end{aligned}$$

the NK Phillips curve (with $\pi_{ss} = 0$) by solving forward:

$$\pi_t = \int_t^\infty e^{-\rho(s-t)}\delta(\rho+\delta)(1+\vartheta)x_s \mathrm{d}s \tag{A.11}$$

The implications of the model for the NK Phillips curve, however, are much richer than suggested by the (log-)linear approximation around steady-state values with $\pi_{ss} = 0$. It also makes it difficult to analyze policy experiments such as changing the inflation target.

A.8. Alternative Taylor principles and stability

To study the stability properties of the dynamic system, the nonlinear system

$$dx_t \equiv f(x_t)dt$$

is approximated by the linear system

$$\frac{d}{dt}x_t = \frac{1}{dt}dx_t = A(x_t - x_{ss})$$

Equivalently, we may study (absolute) deviations from an equilibrium $x_t - x_{ss}$ by defining

$$\frac{d}{dt}(x_t - x_{ss}) = \frac{d}{dt}x_t = A(x_t - x_{ss})$$

such that the Jacobian matrix is identical, or define percentage deviations $\hat{x}_t \equiv x_t/x_{ss} - 1$ for each variable and use $x_t = (1+\hat{x}_t)x_{ss}$ such that for each variable

$$\frac{d}{dt}\hat{x}_t = 1/x_{ss}\frac{d}{dt}x_t = A(x_t - x_{ss})/x_{ss} = A\hat{x}_t$$

with identical Jacobian matrix of the vector function $f(x_t)$.

For illustration, we present the minimal NK model with $s_g = 0$. We compare feedback rule to the partial adjustment rule. In the partial adjustment model, we have:

$$\begin{aligned}
\mathrm{d}i_t &= (\theta\phi_\pi(\pi_t - \pi_{ss}) + \theta\phi_y\hat{y}_t - \theta(i_t - i_{ss}))\mathrm{d}t \\
\Leftrightarrow \mathrm{d}(i_t - i_{ss}) &= (\theta\phi_\pi(\pi_t - \pi_{ss}) + \theta\phi_y\hat{y}_t - \theta(i_t - i_{ss}))\mathrm{d}t \\
\Leftrightarrow \;\; e^{\theta t}\dot{i}_t + e^{\theta t}\theta(i_t - i_{ss}) &= e^{\theta t}\theta\phi_\pi(\pi_t - \pi_{ss}) + e^{\theta t}\theta\phi_y\hat{y}_t \\
\Leftrightarrow \;\; \mathrm{d}(e^{\theta t}(i_t - i_{ss}))/\,\mathrm{d}t &= e^{\theta t}\theta\phi_\pi(\pi_t - \pi_{ss}) + e^{\theta t}\theta\phi_y\hat{y}_t \\
\text{for } \; t_0 \to -\infty \quad \Rightarrow \quad i_t - i_{ss} &= \theta\int_{-\infty}^t e^{-\theta(t-k)}(\phi_\pi(\pi_k - \pi_{ss}) + \phi_y\hat{y}_t)\,\mathrm{d}k,
\end{aligned}$$

which requires $\theta > 0$ or alternatively for the feedback rule model:

$$i_t - i_{ss} = \phi_\pi(\pi_t - \pi_{ss}) + \phi_y(y_t/y_{ss} - 1), \quad \phi_\pi > 1, \; \phi_y \geq 0.$$

Note that for *any* given inflation target, π_{ss}, there exists an equilibrium interest rate, i_{ss}, which leads to an admissible steady state.

A.8.1. Feedback rule

In the feedback rule in the simple NK model we have:

$$i_t - i_{ss} = \phi(\pi_t - \pi_{ss}), \quad \phi > 1$$

or more general, the feedback rule (used in the main text) with an output response:

$$i_t - i_{ss} = \phi_\pi(\pi_t - \pi_{ss}) + \phi_y(y_t/y_{ss} - 1), \quad \phi_\pi > 1,\ \phi_y \geq 0,$$

for example $\phi_\pi \approx 1.5$ and $\phi_y \approx 0.5$ for target rates $\pi_{ss} \approx 0$ (see Woodford, 2001).

To study the properties of the equilibrium points, define $x_t \equiv (y_t, v_t, x_{1,t}, x_{2,t})$ such that

$$f(x_t) \equiv f(y_t, v_t, x_{1,t}, x_{2,t}) = \begin{bmatrix} -(\rho - i_t + \pi_t)\, y_t \\ \delta\,(1 + (1-\varepsilon)\pi_t/\delta)^{-\frac{\varepsilon}{1-\varepsilon}} + (\varepsilon\pi_t - \delta)v_t \\ (\rho + \delta - (\varepsilon - 1)\pi_t)x_{1,t} - 1 \\ (\rho + \delta - \varepsilon\pi_t)\, x_{2,t} - \psi v_t^{\vartheta} y_t^{1+\vartheta} \end{bmatrix}$$

Evaluating the Jacobian matrix at an equilibrium point $x_{ss} = (y_{ss}, v_{ss}, x_{1,ss}, x_{2,ss})$ yields

$$A_1 \quad = \quad \begin{bmatrix} \phi_y & 0 & (1-\phi_\pi)a_2 y_{ss}/x_{1,ss} & -(1-\phi_\pi)a_2 y_{ss}/x_{2,ss} \\ 0 & \varepsilon\pi_{ss} - \delta & -\varepsilon\pi_{ss}v_{ss}/x_{1,ss} & \varepsilon\pi_{ss}v_{ss}/x_{2,ss} \\ 0 & 0 & \rho + \varepsilon a_2 & -(\varepsilon-1)a_2 x_{1,ss}/x_{2,ss} \\ -(1+\vartheta)a_1 x_{2,ss}/y_{ss} & -\vartheta a_1 x_{2,ss}/v_{ss} & \varepsilon a_2 x_{2,ss}/x_{1,ss} & a_1 - \varepsilon a_2 \end{bmatrix}$$

where $a_1 \equiv \rho + \delta - \varepsilon\pi_{ss}$, and $a_2 \equiv \delta + (1-\varepsilon)\pi_{ss}$.

Hence, we may approximate the equilibrium dynamics by

$$\begin{aligned} \mathrm{d}\hat{y}_t &= (i_t - \rho - \pi_t)\mathrm{d}t \\ \mathrm{d}\hat{v}_t &= ((\varepsilon\pi_{ss} - \delta)\hat{v}_t + \varepsilon\pi_{ss}/a_2(\pi_t - \pi_{ss}))\mathrm{d}t \\ \mathrm{d}\hat{x}_{1,t} &= ((\rho + a_2)\hat{x}_{1,t} + (1-\varepsilon)(\pi_t - \pi_{ss}))\,\mathrm{d}t \\ \mathrm{d}\hat{x}_{2,t} &= (a_1\hat{x}_{2,t} - \varepsilon(\pi_t - \pi_{ss}) - (1+\vartheta)a_1\hat{y}_t - \vartheta a_1\hat{v}_t)\mathrm{d}t \end{aligned}$$

where $\pi_t - \pi_{ss} = a_2(x_{2,t}/x_{2,ss} - x_{1,t}/x_{1,ss})$ and $i_t = \phi_y(y_t/y_{ss} - 1) + \phi_\pi(\pi_t - \pi_{ss}) + i_{ss}$ such that the inflation dynamics are:

$$\begin{aligned} \mathrm{d}\pi_t &= \rho(\pi_t - \pi_{ss})\,\mathrm{d}t - (\delta + (1-\varepsilon)\pi_{ss})\pi_{ss}(x_{2,t}/x_{2,ss} - 1)\mathrm{d}t \\ &\quad -\kappa((y_t/y_{ss} - 1) + (v_t/v_{ss} - 1)\vartheta/(1+\vartheta))\mathrm{d}t \end{aligned}$$

in which $\kappa \equiv (\delta + (1-\varepsilon)\pi_{ss})(1+\vartheta)\,(\rho + \delta - \varepsilon\pi_{ss})$.

Around zero-inflation target $\pi_{ss} = 0$ and $i_{ss} = \rho$, the equilibrium dynamics are:

$$
\begin{aligned}
\mathrm{d}\hat{y}_t &= (i_t - \rho - \pi_t)\mathrm{d}t \\
\mathrm{d}\hat{v}_t &= -\delta\hat{v}_t\mathrm{d}t \\
\mathrm{d}\pi_t &= (\rho\pi_t - (1+\vartheta)(\rho+\delta)\delta\hat{y}_t - \vartheta(\rho+\delta)\delta\hat{v}_t)\mathrm{d}t
\end{aligned}
$$

In this first-order approximation, price dispersion is no longer affected by other variables, such that it will always converge. Analyzing equilibrium dynamics will be based on:

$$
\begin{aligned}
\mathrm{d}\hat{y}_t &= (i_t - \rho - \pi_t)\mathrm{d}t \\
\mathrm{d}\pi_t &= (\rho\pi_t - \kappa\hat{y}_t)\mathrm{d}t
\end{aligned}
$$

where $\kappa \equiv (1+\vartheta)\,(\rho+\delta)\,\delta$ and $i_t = i_{ss} + \phi_\pi\pi_t + \phi_y\hat{y}_t$. Sometimes the linearized model around zero inflation target is used to approximate the model around positive inflation targets, $\pi_{ss} > 0$ (e.g., Cochrane, 2017b, eq. (4) with time-varying π_{ss} and ρ).

Based on the reduced system $x = (\hat{y}_t, \pi_t)$ for $\pi_{ss} = 0$, the 2×2 Jacobian matrix reads:

$$
A_1 = \begin{bmatrix} \phi_y & \phi_\pi - 1 \\ -\kappa & \rho \end{bmatrix}
$$

For a unique locally bounded equilibrium we need two positive eigenvalues, for the larger system $\pi_{ss} \neq 0$ we need three positive and one negative eigenvalue.

The Jacobian matrix has $\mathrm{tr}(A_1) = \lambda_1 + \lambda_2 = \phi_y + \rho > 0$ and $\det(A_1) = \rho\phi_y + (\phi_\pi - 1)\kappa$ is positive for $\phi_\pi > 1$, thus both eigenvalues have positive real parts, $\lambda_1\lambda_2 = \det(A_1)$,

$$
\lambda^2 - (\phi_y + \rho)\lambda + \rho\phi_y + (\phi_\pi - 1)\kappa = 0
$$

$$
\lambda_{1,2} = \tfrac{1}{2}(\rho + \phi_y \pm \sqrt{(\phi_y+\rho)^2 - 4(\rho\phi_y + (\phi_\pi - 1)\kappa)})
$$

So the unique locally bounded solution is $\hat{y}_t = 0$ and $\pi_t = \pi_{ss}$ such that $i_t = i_{ss}$.

A.8.2. Partial adjustment

For the partial adjustment model, we need to add the dynamics of the interest rate:

$$
\mathrm{d}(i_t - i_{ss}) = (\theta\phi_\pi(\pi_t - \pi_{ss}) + \theta\phi_y\hat{y}_t - \theta(i_t - i_{ss}))\mathrm{d}t
$$

It relates to Graeve, Emiris, and Wouters (2009), where the Taylor rule has lagged interest rates and potential output (the level of output that would prevail under flexible prices).

To study the properties of the two equilibrium points, define $x_t \equiv (y_t, v_t, x_{1,t}, x_{2,t}, i_t)$

such that

$$f(x_t) \equiv f(y_t, v_t, x_{1,t}, x_{2,t}, i_t) = \begin{bmatrix} -(\rho - i_t + \pi_t)\, y_t \\ \delta\,(1 + (1-\varepsilon)\pi_t/\delta)^{-\frac{\varepsilon}{1-\varepsilon}} + (\varepsilon\pi_t - \delta)v_t \\ (\rho + \delta - (\varepsilon - 1)\pi_t)x_{1,t} - 1 \\ (\rho + \delta - \varepsilon\pi_t)\, x_{2,t} - \psi v_t^{\vartheta} y_t^{1+\vartheta} \\ \theta\phi_\pi(\pi_t - \pi_{ss}) + \theta\phi_y(y_t/y_{ss} - 1) - \theta(i_t - i_{ss}) \end{bmatrix}$$

Evaluating the Jacobian matrix at equilibrium point $x_{ss} = (y_{ss}, v_{ss}, x_{1,ss}, x_{2,ss}, i_{ss})$ yields

$$A_2 = \begin{bmatrix} 0 & 0 & a_2 y_{ss}/x_{1,ss} & -a_2 y_{ss}/x_{2,ss} & y_{ss} \\ 0 & \varepsilon\pi_{ss} - \delta & -\varepsilon\pi_{ss}v_{ss}/x_{1,ss} & \varepsilon\pi_{ss}v_{ss}/x_{2,ss} & 0 \\ 0 & 0 & \rho + \varepsilon a_2 & -(\varepsilon - 1)a_2 x_{1,ss}/x_{2,ss} & 0 \\ -(1+\vartheta)a_1 x_{2,ss}/y_{ss} & -\vartheta a_1 x_{2,ss}/v_{ss} & \varepsilon a_2 x_{2,ss}/x_{1,ss} & a_1 - \varepsilon a_2 & 0 \\ \theta\phi_y/y_{ss} & 0 & -\theta\phi_\pi a_2/x_{1,ss} & \theta\phi_\pi a_2/x_{2,ss} & -\theta \end{bmatrix}$$

where $a_1 \equiv \rho + \delta - \varepsilon\pi_{ss}$, and $a_2 \equiv \delta + (1-\varepsilon)\pi_{ss}$.

Hence, we may approximate the equilibrium dynamics by

$$\begin{aligned} \mathrm{d}\hat{y}_t &= (i_t - \rho - \pi_t)\,\mathrm{d}t \\ \mathrm{d}\hat{v}_t &= ((\varepsilon\pi_{ss} - \delta)\hat{v}_t + \varepsilon\pi_{ss}/a_2(\pi_t - \pi_{ss}))\,\mathrm{d}t \\ \mathrm{d}\hat{x}_{1,t} &= ((\rho + a_2)\hat{x}_{1,t} + (1-\varepsilon)(\pi_t - \pi_{ss}))\,\mathrm{d}t \\ \mathrm{d}\hat{x}_{2,t} &= (a_1\hat{x}_{2,t} - \varepsilon(\pi_t - \pi_{ss}) - (1+\vartheta)a_1\hat{y}_t - \vartheta a_1\hat{v}_t)\mathrm{d}t \\ \mathrm{d}i_t &= (\theta\phi_\pi(\pi_t - \pi_{ss}) + \theta\phi_y\hat{y}_t - \theta(i_t - i_{ss}))\,\mathrm{d}t \end{aligned}$$

where $\pi_t - \pi_{ss} = a_2(x_{2,t}/x_{2,ss} - x_{1,t}/x_{1,ss})$ such that the inflation dynamics are:

$$\mathrm{d}\pi_t = (\rho(\pi_t - \pi_{ss}) - a_2\pi_{ss}\hat{x}_{2,t} - \kappa\hat{y}_t - \vartheta a_1 a_2\hat{v}_t)\mathrm{d}t$$

in which $\kappa \equiv (1+\vartheta)(\rho + \delta - \varepsilon\pi_{ss})(\delta + (1-\varepsilon)\pi_{ss})$.

Around zero-inflation target $\pi_{ss} = 0$ and $i_{ss} = \rho$, the equilibrium dynamics are:

$$\begin{aligned} \mathrm{d}\hat{y}_t &= (i_t - \rho - \pi_t)\,\mathrm{d}t \\ \mathrm{d}\hat{v}_t &= -\delta\hat{v}_t\,\mathrm{d}t \\ \mathrm{d}\pi_t &= (\rho\pi_t - (1+\vartheta)\delta(\rho+\delta)\hat{y}_t - \vartheta\delta(\rho+\delta)\hat{v}_t)\mathrm{d}t \\ \mathrm{d}i_t &= (\theta\phi_\pi\pi_t + \theta\phi_y\hat{y}_t - \theta(i_t - i_{ss}))\,\mathrm{d}t \end{aligned}$$

In this first-order approximation, price dispersion is no longer affected by other variables, such that it will always converge. Analyzing equilibrium dynamics will be based on:

$$\begin{aligned}
\mathrm{d}\hat{y}_t &= (i_t - \rho - \pi_t)\,\mathrm{d}t \\
\mathrm{d}\pi_t &= (\rho\pi_t - \kappa\hat{y}_t)\mathrm{d}t \\
\mathrm{d}i_t &= (\theta\phi_\pi\pi_t + \theta\phi_y\hat{y}_t - \theta(i_t - \rho))\,\mathrm{d}t
\end{aligned}$$

where $\kappa \equiv (1+\vartheta)(\rho+\delta)\delta$.

Based on the reduced system $x_t = (\hat{y}_t, \pi_t, i_t)$ for $\pi_{ss} = 0$, the 3×3 Jacobian matrix reads:

$$A_2 = \begin{bmatrix} 0 & -1 & 1 \\ -\kappa & \rho & 0 \\ \theta\phi_y & \theta\phi_\pi & -\theta \end{bmatrix}$$

For a unique locally bounded equilibrium we need two positive and one negative eigenvalue, for the larger system $\pi_{ss} \neq 0$ we need three positive and two negative eigenvalues.

A.9. Alternative shock processes

Consider the Ornstein-Uhlenbeck (OU) process (7), also used in this paper,

$$\mathrm{d}\log d_t = -\rho_d \log d_t \mathrm{d}t + \sigma_d \mathrm{d}B_{d,t}$$

or

$$\mathrm{d}d_t = -\left(\rho_d \log d_t - \tfrac{1}{2}\sigma_d^2\right) d_t \mathrm{d}t + \sigma_d d_t \mathrm{d}B_{d,t}$$

versus the alternative of a logistic growth process (add stochastics)

$$\mathrm{d}d_t = \rho_d d_t (1 - d_t)\,\mathrm{d}t \tag{A.12}$$

which is the logistic growth model with carrying capacity 1.

The natural (lower) bound is zero and the turning point is 1/2. The process suggests that if the variable is near its carrying capacity, the dynamics are just like those of the OU process, whereas if the variable is near its natural lower bound, the dynamics are similar to exponential growth. The parameter ρ_d measures the half-life. We may scale the model such that it fits our needs to have a prolonged period of persistence of a shock at the beginning and later to revert back to the steady state level geometrically at rate ρ_d such that (similar to the OU process) the higher ρ_d the lower persistence, the smaller ρ_d the more pronounced shocks are smeared out in time. Now consider

$$\mathrm{d}d_t = \rho_d(d_t - \bar{d})(1 - d_t)/(1 - \bar{d})\,\mathrm{d}t \tag{A.13}$$

of which the solution is

$$d_t = \frac{d_{ss} - \bar{d}}{1 + \mathbb{C}e^{-\rho_d t}} + \bar{d}$$

The (unique) steady state value is the solution of

$$0 = \rho_d \left(d_t - \bar{d}\right) \left(1 - \left(d_t - \bar{d}\right) / (d_{ss} - \bar{d})\right) \,\mathrm{d}t$$

where we require that $d_t > \bar{d}$ for all time. Linearizing about d_{ss} yields

$$\mathrm{d}d_t/d_{ss} = -\rho_d(d_t/d_{ss} - 1)\,\mathrm{d}t$$

or

$$\mathrm{d}\hat{d}_t = -\rho_d \hat{d}_t \,\mathrm{d}t$$

It reflects that the logistic growth model for $d_t - \bar{d}$ such that d_t approaches d_{ss}. The variable $d_t - \bar{d}$ is defined between 0 and ∞ with carrying capacity $d_{ss} - \bar{d}$ and turning point at $(d_{ss} - \bar{d})/2$, such that the original variable d_t is defined between $\bar{d}$ and ∞ with turning point at $1 - (d_{ss} - \bar{d})/2$. For $\bar{d} = 0$ we assume logistic growth for d_t, whereas $\bar{d} \to d_{ss}$ squeezes the admissible region lower than the steady state level towards zero, such that $\bar{d}$ denotes the lower bound for d_t (typically zero in the OU case). Any (negative) shock larger than $1 - (d_{ss} - \bar{d})/2$ induces quite different dynamics, staying there for some time before returning to the steady state level (cf. Figure C.42). This effect only shows up in the nonlinear version of the model. While the logistic model looks very much like an exponential model in the beginning, around the steady state value, the linearized dynamics are the same as for the Ornstein-Uhlenbeck process. It is important to understand that the linear model thus will not be able to replicate those dynamics.

A.10. The yield curve

Consider a nominal (zero-coupon) bond with unity payoff at maturity $N > t$. We obtain the equilibrium price in the perfect-foresight model as

$$P_t^{(N)} = (m_{t+N}/m_t)e^{-\int_t^{t+N} \pi_s ds}, \tag{A.14}$$

where π_t denotes inflation. Inserting the SDF, $m_t = e^{-\rho t}V_K = e^{-\rho t}d_t/c_t$ we obtain that

$$\begin{aligned} P_t^{(N)} &= e^{-\rho N}((V_K)_{t+N}/(V_K)_t)e^{-\int_t^{t+N} \pi_s ds} \\ &= e^{-\rho N}(c_t/c_{t+N})(d_{t+N}/d_t)e^{-\int_t^{t+N} \pi_s ds} \end{aligned}$$

The yield of a bond is the fictional interest rate that justifies the quoted price, assuming that the bond does not default, hence

$$P_t^{(N)} = 1/\left[Y_t^{(N)}\right]^N$$

or

$$Y_t^{(N)} = 1/\left[P_t^{(N)}\right]^{1/N}$$

such that the log price satisfies

$$y_t^{(N)} = -(1/N)p_t^{(N)}$$

Hence, in our model we obtain the log yield with maturity $N > t$ as

$$y_t^{(N)} = \rho + (1/N)\log(c_{t+N}/c_t) - (1/N)\log(d_{t+N}/d_t) + (1/N)\int_t^{t+N} \pi_s ds$$

B. Tables and Figures

Figure 1: US federal funds rate, 10-year treasury rate and inflation rate
In this figure we show time series plots of the US Effective Federal Funds Rate (Fed Funds), the 10-Year Treasury Constant Maturity Rate (10Y Govt), the Consumer Price Index, seasonally adjusted (Core CPI), all at the monthly frequency. All series are obtained from the Federal Reserve Bank of St. Louis Economic Dataset (FRED). The sample runs from January, 1990, through June, 2017.

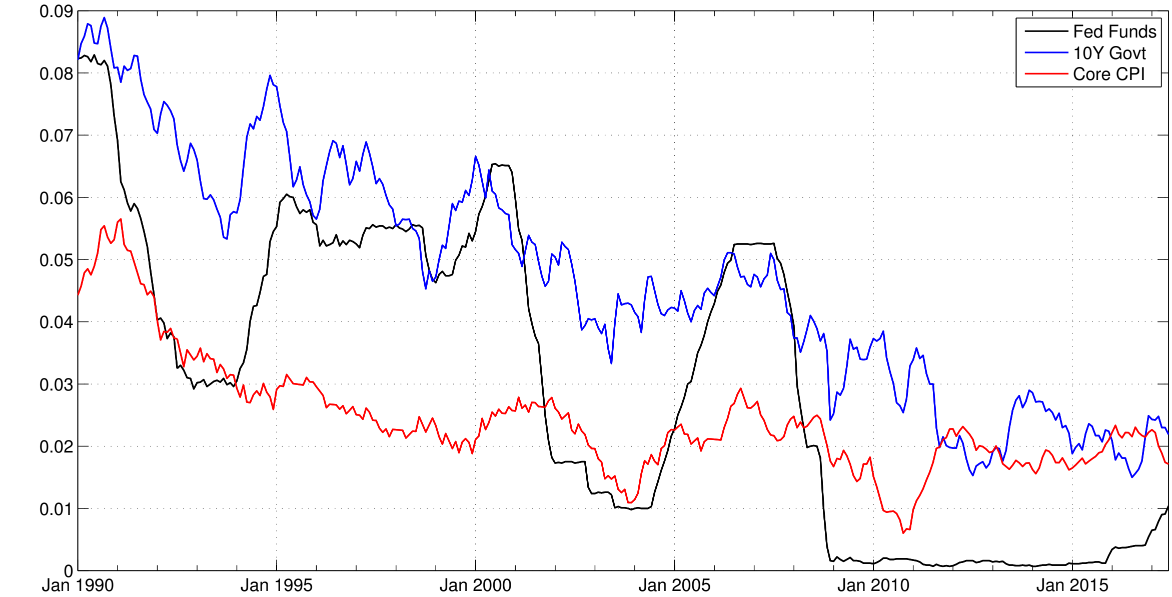

Figure 2: Implied 10-year treasury rate and natural rate
In this figure we show time series plots of the model-implied 10-Year Treasury Rate (Implied 10Y Govt) and the model-implied 'natural rate' using the simple NK model around $\pi_{ss} = 0$, when matching the observed US Effective Federal Funds Rate (Fed Funds) and the Consumer Price Index, seasonally adjusted, all at the monthly frequency. The sample runs from January, 1990, through June, 2017.

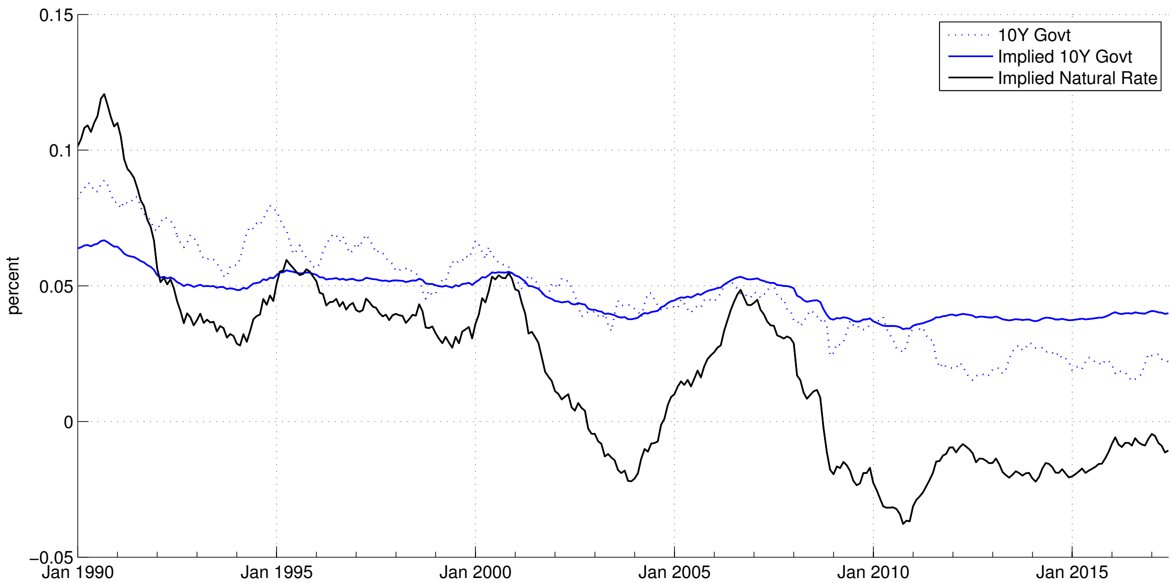

Figure 3: Solution of the linearized NK model with partial adjustment

In this figure we show (from left to right) the output gap, and the inflation rate as a function of the (initial) interest rate for a parameterization $(\rho, \kappa, \phi, \theta, \pi_{ss}) = (0.01, 0.8582, 4, 0.5, 0)$.

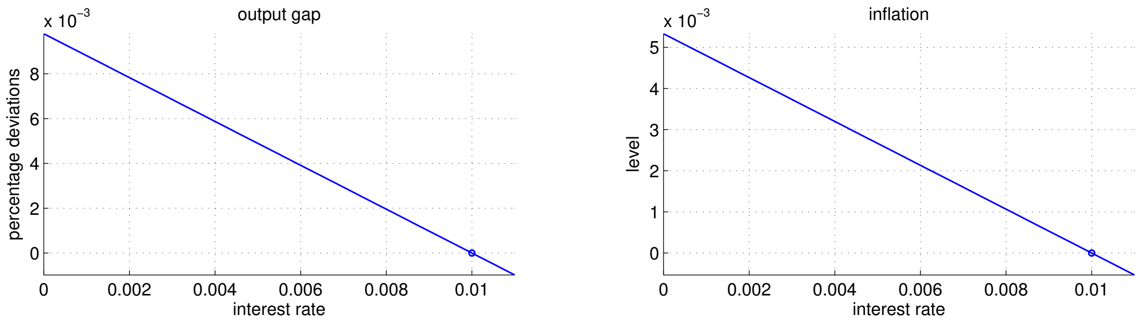

Figure 4: Solution of the minimal NK model with partial adjustment

In this figure we show (from left to right) the output gap, and the inflation rate as a function of the (initial) interest rate in the minimal model (blue solid), in the linearized model (dashed), and in the simple model (around $\pi_{ss} = 0$, dotted) for a parameterization $(\rho, \kappa, \phi_\pi, \phi_y, \theta, \pi_{ss}) = (0.03, 0.8842, 4, 0, 0.5, 0.02)$, .

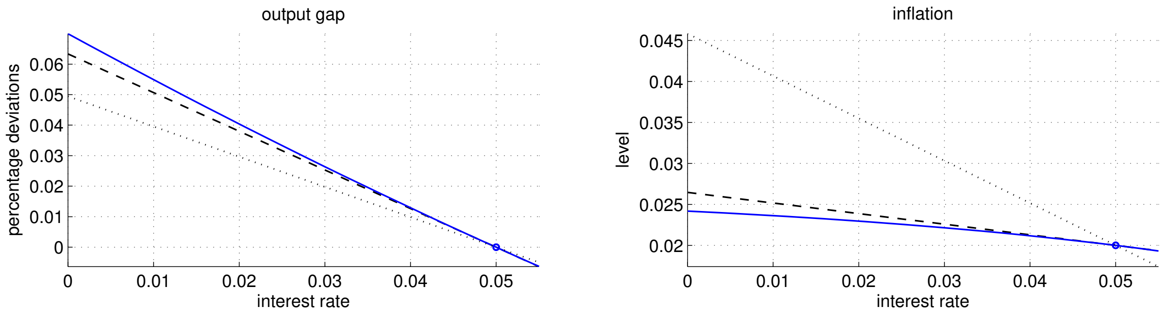

Figure 5: Responses to monetary policy shocks (temporary and permanent)

In this figure we show (from left to right, top to bottom) the simulated responses for unexpected shocks to the (initial) interest rate (-0.025), and the inflation target rate (-0.005), with effects for the output gap, the inflation rate, and the level and slope of the interest rate (blue solid), and the no-target rate shock scenario in the simple model (black dashed, $\pi_{ss} = 0.02$).

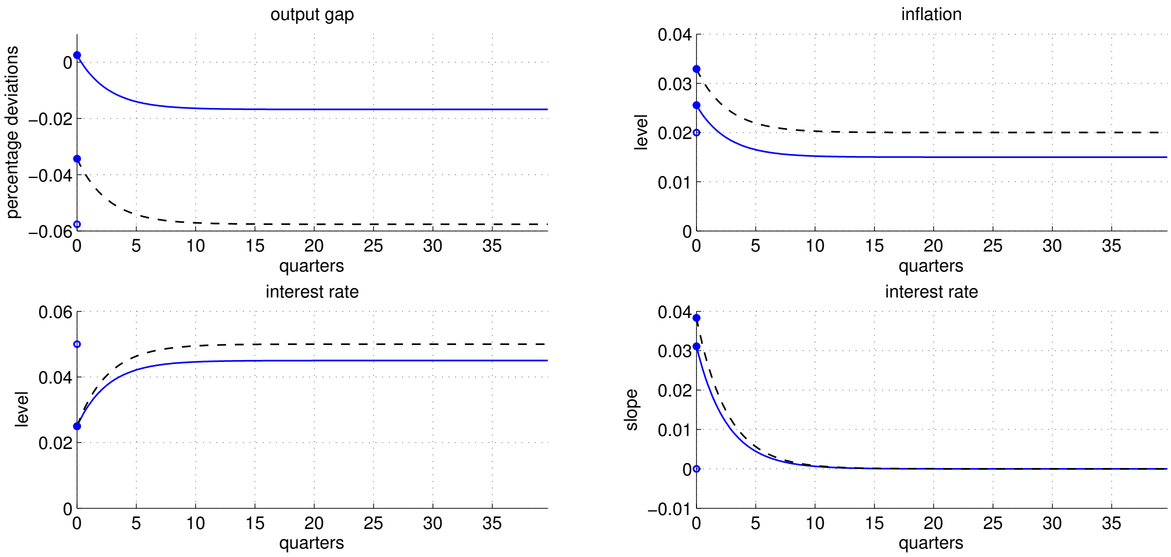

Figure 6: Simulated shock to interest rate and target rate (2007-2011), macro dynamics
In this figure we show (from left to right, top to bottom) the simulated responses for unexpected shocks to the (initial) interest rate (-0.0475), the inflation target rate (-0.02), and preferences (-0.1), with effects for the output gap, the inflation rate, and the level and the slope of the interest rate (blue solid), the no-target rate shock scenario (black dashed, $\pi_{ss} = 0.02$), and the pre-shock scenario (dotted).

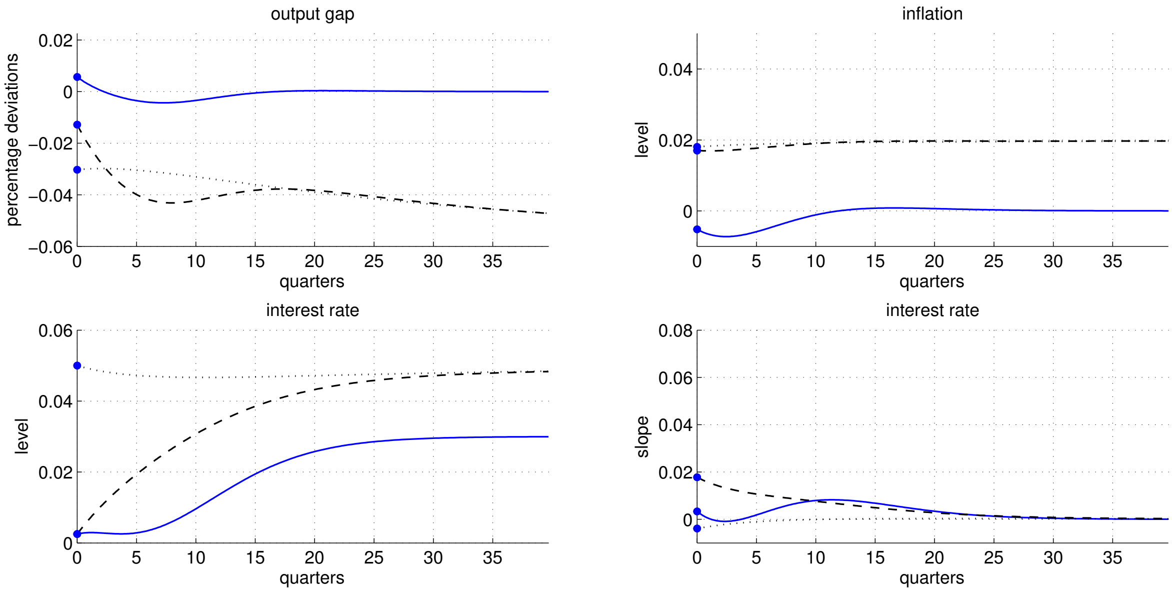

Figure 7: Simulated shock to interest rate and target rate (2007-2011), yield curve
In this figure we show (from left to right) the yield curve response to unexpected shocks to the (initial) interest rate (-0.0475), the inflation target rate (-0.02), and preferences (-0.1), with effects for the nominal and real yields (blue solid), the no-target rate shock scenario (black dashed, $\pi_{ss} = 0.02$), and the pre-shock scenario (dotted).

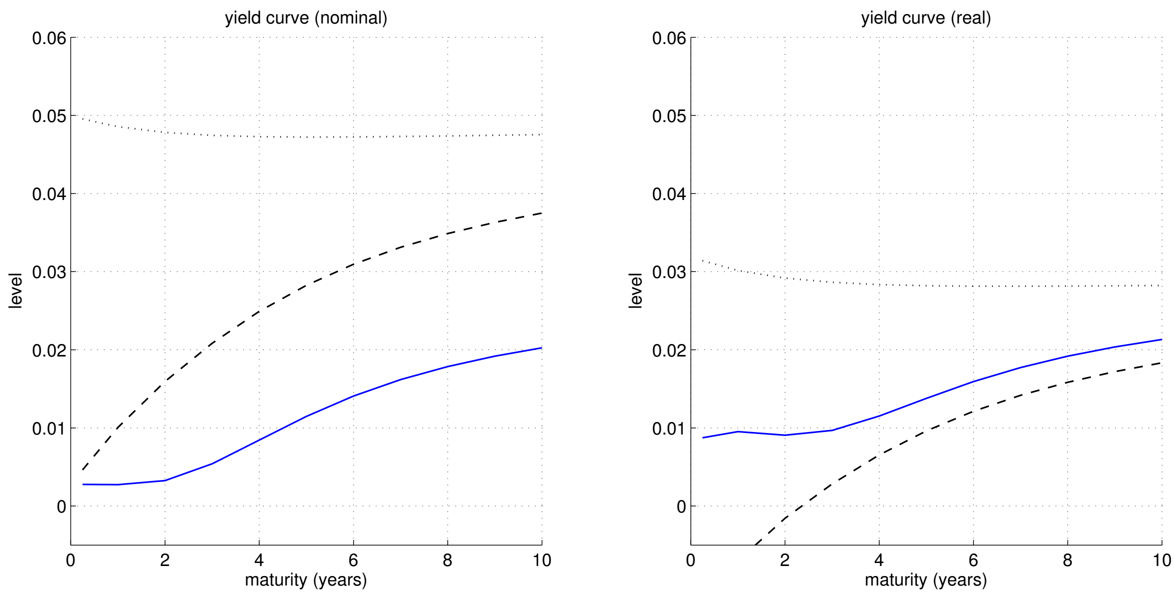

Table 1: Summary of the solution algorithm in the policy function space

Step 1	(Initialization)	Provide an initial guess for the unknown derivatives for a given set of collocation nodes and basis functions.
Step 2	(Solution)	Compute the optimal value of the controls for the set of nodal values for the state variables.
Step 3	(Update)	Update the consumption function derivatives.
Step 4	(Iteration)	Repeat Steps 2 and 3 until convergence.

Table 2: Parameterization

ϑ	1	Frisch labor supply elasticity
ρ	0.03	subjective rate of time preference, $\rho = -4 \log 0.9925$
ψ	1	preference for leisure
δ	0.65	Calvo parameter for probability of firms receiving signal, $\delta = -4 \log 0.85$
ε	25	elasticity of substitution intermediate goods
s_g	0.2	share of government consumption
ρ_d	0.4214	autoregressive component preference shock, $\rho_d = -4 \log 0.9$
r_A	0.4214	autoregressive component technology shock, $r_A = -4 \log 0.9$
r_g	0.4214	autoregressive component government shock, $r_g = -4 \log 0.9$
σ_d	0.02	variance preference shock
σ_A	0.02	variance technology shock
σ_g	0.02	variance government shock
σ_m	0.02	variance monetary policy shock
ϕ_π	4	inflation response Taylor rule
ϕ_y	0	output response Taylor rule
θ	0.5	interest rate response Taylor rule
π_{ss}	0.02	inflation target rate

Table 3: Data and NK model predictions around $\pi_{ss} = 0$

	Data			NK model (transitory)				NK model (both)			
	FF	10Y	CPI	FF	10Y	TR	CPI	FF	10Y	TR	CPI
2001-01	6%	5%	2.5%	6%	5%	2%	1.5%	6%	5%	2%	1.5%
2003-12	1%	3.5%	1%	1%	5%	2%	4%	1%	3%	0%	1%
2007-09	5%	4.5%	2%	5%	5%	2%	2%	5%	5%	2%	2%
2009-01	0.25%	2.5%	2%	0.25%	4.5%	2%	4.5%	0.25%	3%	0%	1.5%
2010-10[a]	0.25%	2.5%	0.5%	0.25%	4.5%	2%	4%	0.25%	2.5%	0%	1%
2011-06[a]	0%	3%	1.5%	0%	4%	2%	1%	0%	4%	2%	1%
2004-06	1%	4.5%	2%	1%	4.5%	2%	4%	1%	4.5%	2%	4%
2005-02[a]	2.5%	4%	2.5%	2.5%	4%	2%	1%	2.5%	4%	2%	1%

[a)] The predictions include (negative) shocks to the natural rate along with the monetary policy shocks.

Table 4: Data and NK model predictions (full model)

	Data			NK model (transitory)				NK model (both)			
	FF	10Y	CPI	FF	10Y	TR	CPI	FF	10Y	TR	CPI
2001-01	6%	5%	2.5%	6%	5%	2%	2%	6%	5%	2%	2%
2003-12	1%	3.5%	1%	1%	4.5%	2%	2.5%	1%	3%	0%	1.5%
2007-09	5%	4.5%	2%	5%	5%	2%	2%	5%	4.5%	2%	2%
2009-01	0.25%	2.5%	2%	0.25%	4.5%	2%	2.5%	0.25%	3%	0%	2%
2010-10[a]	0.25%	2.5%	0.5%	0.25%	4%	2%	2%	0.25%	2%	0%	0%
2011-06[a]	0%	3%	1.5%	0%	3%	2%	1%	0%	3%	2%	1%
2004-06	1%	4.5%	2%	1%	4.5%	2%	2.5%	1%	4.5%	2%	2.5%
2005-02[a]	2.5%	4%	2.5%	2.5%	4%	2%	2%	2.5%	4%	2%	2%

[a)] The predictions include (negative) shocks to the natural rate along with the monetary policy shocks.